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Relativistic elasticity of dissipative media and its wave propagation modes

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Abstract. A phenomenological, general relativistic theory of dissipative elastic solids whose equations form a hyperbolic system is proposed. The non-stationary transport equations for dissipative fluxes containing new cross-effect terms, as required by compatibility with irreversible thermodynamics, have been adopted. As opposed to some conventional theories which are parabolic and predict instantaneous propagation of wavefronts, the theory formulated, consisting of 14 partial differential equations (in the case of special relativity), of total order 17, is hyperbolic and predicts, for all existing propagation modes, finite front speeds. The complete system of special relativistic propagation modes of an elastic solid is determined from the linearised equations. There are four mutually distinct non-trivial propagation modes, two for longitudinal waves and two for transverse waves. If the rigidity modulus decreases to zero one obtains as a special case the normal modes for fluid according to the theory of Müller and Israel. Weber's equation is briefly discussed too.

1. Introduction

A clear review of the reasons for the development of relativistic elasticity has been given by Hernandez (1970): 'By far, most of the past work in relativity has been concerned with either the vacuum or fluid-type materials. Yet there are several reasons why elasticity theory, and more generally, non-fluid theories should be well understood. (*a*) Elastic bodies do exist. Even though relativistic effects are small, the theory should still allow for these solutions. (*b*) Under 'abnormal' conditions, matter requiring relativistic description may possess non-fluid properties. For example Misner (1968) has pointed out that in early stages of big-bang cosmology, for temperatures of 10^5–10^{10} K, the collisionless neutrino radiation possesses properties similar to those of an elastic solid. It is also possible that the superdense materials of the even earlier stages of the big-bang model or the neutron star interiors might possess non-fluid properties. (*c*) Static non-fluid bodies can be aspherical (in contrast to fluid bodies) and hence can be of interest in studying aspherical effects in general relativity. For example, only non-fluid bodies can serve as sources for the static axisymmetric Weyl metrics.'

For current terrestrial physics, the theory of methods for measurement and production of gravitational waves seems to be of importance. An elastic body constitutes the most simple antenna for reception of gravitational radiation (see, e.g., Papapetrou 1972 or Dyson 1969). The elastic vibrations generated by the absorption of gravitational waves (a problem which was recently intensively studied by Weber 1961 and later) are attenuated by the damping due to dissipation of energy in the elastic body. To investigate such problems a knowledge of the relativistic theory for the dissipative solid is required.

It is evident that, due to a lack of literature on this subject, the first step to take is to establish, in some simple manner, the theory of the dissipative elastic solid, as Weber (1961) was forced to do. However, in his treatment the dissipation is taken into account simply by the inclusion of a frictional damping force which is proportional to the velocity of vibrating particles. For such a description, thermodynamics does not exist and the resulting wave equation leads to a dispersion dependence which is in contradiction with classical experiments both in fluids and in solids (see Kranyš 1977, to be referred to as I). (Usually, dissipation causes an increase in the phase speeds of the waves; but this is not true in Weber's (1961) description.) Such an approach could be adequate, perhaps, for one- or two-dimensional bodies but not for a three-dimensional solid body, as pointed out by Maugin (1974a) who proposed a relativistic constitutive equation for the Kelvin–Voigt viscoelastic solid. The resulting equation for strain propagation, according to Maugin, is compatible with conventional stationary thermodynamics, and is free of the defect in Weber's equation and therefore much more realistic. However, besides the simplification due to the neglect of heat conduction, Maugin's equation is parabolic which means that an infinite front speed for propagating waves is predicted.

In accordance with the theory of relativity we believe that all possible field and/or material phenomena can propagate only with some finite signal velocity not exceeding the velocity of light *in vacuo*. From the mathematical point of view this requirement can be fulfilled only if the system of partial differential equations describing the appropriate physical phenomena is hyperbolic. However, this is not true for a thermodynamically stationary theory: to the category of stationary theories belong, for example, the phenomenological fluid theory of Eckart (1940) (see also Grot and Eringen 1966). This theory, and its various slight modifications, apply only the minimum necessary generalisations in introducing dissipation, i.e. viscous stress, bulk stress and heat flux, by means of Navier–Newton and Fourier transport equations which are evidently stationary.

This difficulty with stationary theories led some authors to propose more satisfactory mathematical systems which must evidently be non-stationary and hyperbolic at the same time. In classical physics, where a similar problem also existed, propositions of non-stationary heat transport equations emerged around 1948 (Cattaneo and others—see the story in I). In relativity theory, the first non-stationary heat transport equations, i.e. amended with a relaxation term, were proposed by Kranyš (1964, 1966a) who later (Kranyš 1966b, 1967) gave the non-stationary transport equations for viscosity. The general proof of the hyperbolicity of the theory, using the modified heat transport equation, was given by Mahjoub (1969, 1971a, b) and also by Boillat (1971, 1972) who studied various alternatives to hyperbolic systems. Müller (1966, 1967) proposed phenomenological transport equations in a fluid including other terms besides the previously mentioned relaxation terms, and he also showed, strictly according to non-equilibrium thermodynamics, that retaining systematically all second-order terms in the entropy balance equation, the more accurate form of the transport equations for dissipative fluxes can be predicted. The same result as that of Müller for a relativistic fluid continuum was also arrived at recently by Israel (1976). Stewart (1971, 1974) used the ideas of Marle (1969), Kranyš and Müller in his studies too. The attempt to generalise the constitutive transport equations in order to obtain a hyperbolic theory based on 'constitutive axioms' was made by Müller (1969, 1972), Maugin (1973a, 1974a) and Barrabes (1975).

We do not want to use the direct arguments of kinetic theory in this paper dealing with continuum theory, but we will use phenomenological non-stationary thermodynamics which is consistent with kinetic theory, *as it must be*, in order for continuum theory as applied to fluids to agree with kinetic theory.

No doubt recent developments in kinetic theory (Grad 1949) have clarified the form of the transport equations for heat flux and viscosity tensor. Even if this was done only for a dilute gas, it gave a firm basis for the revision of the transport equations. Müller was able, within his approximations, to justify each term of the transport equations, both on the basis of phenomenological and kinetic considerations, at least in the classical version. In the relativistic version, the homologue of Grad's 13-moments method is Chernikov's (1962) 13-moment system which does not include bulk viscosity effects (and which was extended to the hyperbolic system by Kranyš 1972) and the 14-moment method first developed by Marle (1969). The propagation modes according to all those three theories, in the linearised case, are given in Kranyš (1972, 1975, 1976). This study reveals that for the last two theories there are three distinct non-trivial propagation modes in a gas, two for longitudinal waves and one for transverse waves, all well below the speed of light *in vacuo*. In particular the ultra-relativistic wavefront speeds according to Marle's 14-moment theory are respectively $0{\cdot}577c$, $0{\cdot}775c$, and $0{\cdot}447c$. The number $0{\cdot}775c$ is found also in Stewart (1971) who seems to have used a modification of Marle's theory, but its interpretation and derivation is not clear. A method of applying the Müller equations to non-relativistic elasticity theory has recently been proposed (I). Additionally, Müller has proposed a relativistic fluid theory, which has been confirmed and complemented by Israel (1976) (see also Israel and Stewart 1976). The purpose of the present paper is to incorporate these ideas into the relativistic theory for an isotropic, elastic, dissipative continuous medium. As we will see below, this requires the inclusion of some cross-effect coupling terms in the transport equations and modification of the form of some thermodynamic equations such as, e.g., the Gibbs equation.

2. Formulation of phenomenological theory for a dissipative solid

We will use as a base the Eckart (1940) type theory which utilises a particle frame (i.e. in the rest frame where particles are at rest in contrast to the energy frame where flux of energy appears to be zero, which is the case in the Landau and Lifshitz type theory. The symmetrical energy–momentum tensor is defined†

$$T^{\alpha\beta}=\rho\epsilon u^{\alpha}u^{\beta}+(2/c)q^{(\alpha}u^{\beta)}+w^{\alpha\beta};\qquad u_{\alpha}u^{\alpha}=-1,\qquad (\alpha,\beta=0,1,2,3)\qquad(2.1)$$

where ρ is the mass density , ϵ the specific internal energy, u^{α} the four-velocity vector, q^{α} the heat flux vector, and $w^{\alpha\beta}$ the total stress tensor; we also use the 'orthogonality' condition with respect to u^{α}:

$$q^{\alpha}u_{\alpha}=0,\qquad w^{\alpha\beta}u_{\beta}=0,\qquad (w^{\alpha\beta}=w^{(\alpha\beta)}).\qquad(2.2)$$

Vectors and tensors fulfilling such conditions are called transverse (with respect to the world line) or 'three-dimensional' quantities and occasionally are marked as

$$q^{\alpha}=\overset{\perp}{q}{}^{\alpha}\equiv\overset{\perp}{g}{}^{\alpha}_{\beta}q^{\beta},\qquad w^{\alpha\beta}=[w^{\alpha\beta}]_{\perp}\equiv\overset{\perp}{g}{}^{\alpha}_{\gamma}\overset{\perp}{g}{}^{\beta}_{\delta}w^{\gamma\delta},\quad\ldots\quad \overset{\perp}{g}{}^{\alpha}_{\beta}\equiv g^{\alpha}_{\beta}+u^{\alpha}u_{\beta}\qquad(2.3)$$

† Parentheses around a set of indices denote symmetrisation; e.g. $q^{(\alpha}u^{\beta)}=\frac{1}{2}(q^{\alpha}u^{\beta}+q^{\beta}u^{\alpha})$.

where $\overset{\perp}{g}{}^{\alpha}_{\beta} = [g^{\alpha}_{\beta}]_{\perp}$ is the metric tensor on the three-dimensional hypersurface locally orthogonal to u^{α}. We shall use also the following rule. If any tensor $X^{\alpha\beta\ldots}_{\ldots}$ is contracted with some transverse tensor $Y^{\ldots}_{\alpha\beta\ldots} \equiv [Y^{\ldots}_{\alpha\beta\ldots}]_{\perp}$ then the following relation holds:

$$X^{\alpha\beta\ldots}_{\ldots}[Y^{\ldots}_{\alpha\beta\ldots}]_{\perp} = [X^{\alpha\beta\ldots}_{\ldots}]_{\perp}[Y^{\ldots}_{\alpha\beta\ldots}]_{\perp}. \tag{2.4}$$

Further we shall also use the following symbols and relations

$$\overset{\perp}{\nabla}_{\alpha} \equiv \overset{\perp}{g}_{\alpha\beta}\nabla^{\beta}, \qquad \mathscr{D} \equiv u^{\alpha}\nabla_{\alpha}, \qquad \nabla_{\alpha} = -u_{\alpha}\mathscr{D} + \overset{\perp}{\nabla}_{\alpha},$$
$$\langle [Z_{\alpha\beta}]_{\perp}\rangle \equiv [Z_{(\alpha\beta)}]_{\perp} - \tfrac{1}{3}\overset{\perp}{g}_{\alpha\beta}[Z^{\lambda}_{\lambda}]_{\perp} \tag{2.5}$$

where $\langle [Z_{(\alpha\beta)}]_{\perp}\rangle$ is called the deviator (from the spherical tensor $\tfrac{1}{3}\overset{\perp}{g}_{\alpha\beta}[Z^{\lambda}_{\lambda}]_{\perp}$) of the tensor $[Z_{\alpha\beta}]_{\perp}$. $\mathscr{L}$ indicates the Lie derivative with respect to the four-vector field u^{α}. (Some basic formulae are gathered in appendix 1.)

The fundamental equations of the theory are the conservation laws for mass, linear momentum and energy (i.e. first principle of thermodynamics) in its standard form, namely

$$\nabla_{\alpha}(\rho c u^{\alpha}) = 0 \qquad \text{or} \qquad c\mathscr{D}\rho + \rho\nabla_{\alpha}cu^{\alpha} = 0, \tag{2.6}$$

$$\overset{\perp}{g}{}^{\sigma}_{\beta}\nabla_{\alpha}T^{\alpha\beta} \equiv \frac{\rho\epsilon}{c^2}c\mathscr{D}cu^{\sigma} + \nabla_{\alpha}w^{\alpha\sigma} - u^{\sigma}w^{\alpha\beta}\overset{\perp}{\nabla}_{(\alpha}u_{\beta)} + \frac{1}{c^2}(c\mathscr{D}q^{\sigma} - u^{\sigma}q^{\beta}\mathscr{D}cu_{\beta}) = 0 \tag{2.7}$$

$$-cu_{\beta}\nabla_{\alpha}T^{\alpha\beta} \equiv \rho c\mathscr{D}\epsilon + w^{\alpha\beta}\overset{\perp}{\nabla}_{(\alpha}cu_{\beta)} + \nabla_{\alpha}q^{\alpha} + \frac{1}{c^2}q^{\alpha}c\mathscr{D}cu_{\alpha} = 0. \tag{2.8}$$

If, for the moment, we consider only special relativity theory and only reversible changes, we can set $q^{\alpha} = 0$, and $\nabla^{\alpha} = \partial^{\alpha}$ and so we have five equations for the eleven unknowns ρ, u^{α}, $w^{\alpha\beta}$, ϵ. Thus we need in this case to express the six components of the stress tensor $w^{\alpha\beta}$ via the remaining five basic unknowns (ρ, u^{α}, ϵ or T) in order to obtain a determinate theory. This relation (a constitutive equation) is evidently the stress–strain relation.

A dissipation-free (i.e. reversible) theory of elasticity in general relativity has been proposed by several authors. Due to the fact that the concept of rate of strain $cs_{\alpha\beta}$ is easy to put into mathematical form in general relativity in comparison with the concept of strain $e_{\alpha\beta}$, Synge (1959), and also later on with some modification, Bennoun (1964, 1965), proposed the basic constitutive equation of elasticity which is a stress–strain relation (Hooke's law) in terms of rate of change of stress $c\mathscr{L}\theta_{\alpha\beta}$ and strain $cs_{\alpha\beta}$. Later Rayner (1963) proposed stress–strain formulae analogous to Hooke's law $\theta_{\alpha\beta} = \bar{C}^{\gamma\delta}_{\alpha\beta}e_{\gamma\delta}$. The direct definition of local strain $e_{\alpha\beta}$ in general relativity is due to Hernandez (1970):

$$e_{\alpha\beta} = \tfrac{1}{2}(\overset{\perp}{g}_{\alpha\beta} - \overset{\perp 0}{g}_{\alpha\beta}); \qquad \mathscr{L}\overset{\perp 0}{g}_{\alpha\beta} = 0 \qquad (\overset{\perp 0}{g}_{\alpha\beta}u^{\beta} = 0) \tag{2.9}$$

where $\overset{\perp 0}{g}_{\alpha\beta}$ is a time-independent metric tensor which describes a flat three-dimensional space of the natural state, and $\overset{\perp}{g}_{\alpha\beta}$ is the spatial metric of the body seen by a local, co-moving observer whose coordinate system is referred to as the local distorted rest frame (LDRF), where 'distorted' refers to the possibility of a non-zero strain, $\overset{\perp}{g}_{\alpha\beta} \neq \overset{\perp 0}{g}_{\alpha\beta}$, while the term 'rest frame' means that $g_{0i} = 0$. For the definition of relativistic strains, see also Maugin (1971) and especially Carter and Quintana (1972). Hernandez also claims that the theories of Synge and Bennoun have nothing to say about static problems. Independently of the veracity of this statement we are allowed to use the concept of rate of strain $cs_{\alpha\beta}$ or absolute strain $e_{\alpha\beta}$ (which are mutually compatible), our concern being the treatment of dissipation processes, which is always a dynamical phenomenon. (In the static state there is no dissipation.)

Let us define (to conform with Synge and Bennoun) the rate of strain tensor

$$cs_{\alpha\beta} \equiv [\nabla_{(\alpha}cu_{\beta)}]_{\perp} = \overset{\perp}{\nabla}_{(\alpha}cu_{\beta)}; \qquad s_{\alpha\beta}u^{\beta} = 0. \tag{2.10}$$

Taking the Lie derivative of $e_{\alpha\beta}$ (see (2.9)) and making use of (2.9) and (A.4) we obtain

$$c\mathscr{L}e_{\alpha\beta} = \tfrac{1}{2}c\overset{\perp}{\mathscr{L}}g_{\alpha\beta} = \overset{\perp}{\nabla}_{(\alpha}cu_{\beta)} = cs_{\alpha\beta}, \qquad (\mathscr{L}e_{\alpha\beta} = [\mathscr{L}e_{\alpha\beta}]_{\perp}) \tag{2.11}$$

which shows the compatibility of both concepts $cs_{\alpha\beta}$ and $e_{\alpha\beta}$; $cs_{\alpha\beta}$ having the advantage of being a 'three-dimensional' tensor and also a convenient function of u_{β} which is one of the basic variables of the theory.

2.1. Constitutive equations for dissipative transport

For fluids the dissipative effects are due to viscosity and thermal conduction and these effects can be investigated analytically by starting from a phenomenological or kinetic theory and using suitable equations. In spite of some differences between fluids and solids (the behaviour in the latter being more complex and varying considerably with the nature of solid), phenomenological theories taking into account dissipation in terms of internal friction (i e. viscosity) and heat conduction are generally used. We want to use the same principle in our development, but instead of employing Fourier's law and the Meyer–Kelvin–Voigt stress–strain relation, both of which imply a parabolic system, we intend to adapt and use the appropriate transport equations converted into a non-stationary form by including relaxation and other terms in order that those generalisations be compatible with more exact non-stationary thermodynamics and which forms a hyperbolic system together with the conservation equations.

In order to establish such a theory let us make the following postulates, which are similar to those used in I when the propositions of the corresponding classical theory are given in more detail.

(i) First we assume (as in Voigt 1892) that the total stress tensor $w^{\alpha\beta}$ in a solid can be expressed as a sum of the reversible or recoverable part of the stress $\theta^{\alpha\beta}$ and the irreversible or dissipative part of the stress $\Pi^{\alpha\beta}$:

$$w^{\alpha\beta} = \theta^{\alpha\beta} + \Pi^{\alpha\beta}; \qquad \theta^{\alpha\beta}u_{\beta} = 0, \quad \Pi^{\alpha\beta}u_{\beta} = 0, \quad \theta^{\alpha\beta} = \theta^{(\alpha\beta)}, \quad \Pi^{\alpha\beta} = \Pi^{(\alpha\beta)}. \tag{2.12}$$

As a consequence of this postulate, $\theta^{\alpha\beta}$, as a fully reversible perfectly elastic stress, possesses the elastic potential ψ' (the Helmholtz free energy) which is the reversible part of $\psi \equiv \epsilon - \eta T = \psi' - T\Delta\eta$ (see (2.36)). In the following, for the sake of clarity, we limit ourselves only to small, so called infinitesimal deformations (i.e. linear stress–strain relations) although such a limitation is not necessary for the development of our thesis†.

† More precisely one could keep the basic conservation equations (2.6)–(2.8) non-linear, or linearise them (but only after the transport equations have been deduced!). However the transport equations are linear in dissipative fluxes.

(In the general (non-linear) case one proceeds as in Maugin (1974a)†.) With this in mind, we can write‡

$$\theta^{\alpha\beta} = -\rho\left[\left(\frac{\partial\psi'}{\partial e_{\alpha\beta}}\right)_T\right]_\perp; \qquad \psi' \equiv \epsilon' - \eta' T, \qquad \eta' = -\left(\frac{\partial\psi'}{\partial T}\right)_e. \tag{2.13}$$

We assume, for perfectly elastic deformations, that the stress and ψ' depend only upon the two state variables and, i.e.

$$\theta^{\alpha\beta} = \theta^{\alpha\beta}(e_{\gamma\delta}, T), \qquad \psi' = \psi'(e_{\alpha\beta}, T). \tag{2.14}$$

Expanding $\psi(e_{\alpha\beta}, T)$ in a Taylor series about the reference point $(0, T_0)$ (writing simply ψ_0 for $\psi(0, T_0)$ etc) we obtain

$$\psi' - \psi'_0 = -\frac{1}{\rho_0} e_{\alpha\beta} A^{\alpha\beta} + (T - T_0)\psi^{(1)} + \frac{1}{2\rho_0} e_{\alpha\rho} e_{\gamma\delta} B^{\alpha\beta\gamma\delta} + \frac{1}{\rho_0} e_{\alpha\beta}(T - T_0) C^{\alpha\beta} + \tfrac{1}{2}(T - T_0)^2 \psi^{(2)} + \ldots \tag{2.15}$$

where the coefficients

$$A^{\alpha\beta} = -\rho_0\left(\frac{\partial\psi'}{\partial e_{\alpha\beta}}\right)_0, \qquad B^{\alpha\beta\gamma\delta} = \rho_0\left(\frac{\partial^2\psi'}{\partial e_{\alpha\beta}\partial e_{\gamma\delta}}\right)_0, \qquad C^{\alpha\beta} = \rho_0\left(\frac{\partial^2\psi'}{\partial T\partial e_{\alpha\beta}}\right)_0,$$
$$\psi^{(1)} = \left(\frac{\partial\psi'}{\partial T}\right)_0, \qquad \psi^{(2)} = \left(\frac{\partial^2\psi'}{\partial T^2}\right)_0 \tag{2.16}$$

are constants, therefore

$$\mathscr{L}A^{\alpha\beta} = 0, \qquad \mathscr{L}B^{\alpha\beta\gamma\delta} = 0, \qquad \mathscr{L}C^{\alpha\beta} = 0,$$
$$\mathscr{L}\psi^{(1)} = 0, \qquad \mathscr{L}\psi^{(2)} = 0. \tag{2.17}$$

From (2.13) and (2.15), for linearly thermo-elastic bodies (consistent with an infinitesimal deformation), one deduces

$$-\theta^{\alpha\beta} \equiv \rho\left[\left(\frac{\partial\psi'}{\partial e_{\alpha\beta}}\right)_T\right]_\perp = -[A^{\alpha\beta}]_\perp + [B^{\alpha\beta\gamma\delta} e_{\gamma\delta}]_\perp + (T - T_0)[C^{\alpha\beta}]_\perp \tag{2.18}$$

† In the general (non-linear) case the following equation holds:

$$\theta^{\alpha\beta} = -\rho\left[\left(\frac{\partial\psi'}{\partial e_{\alpha\beta}}\right)_T - 2e_\mu^{(\alpha}\left(\frac{\partial\psi'}{\partial e_{\beta)\mu}}\right)_T\right]_\perp.$$

If we consider only infinitesimal strains, i.e. $(e^\alpha_\beta e^\beta_\alpha)^{1/2} \ll 1$ the second term in this equation drops out and we obtain

$$\theta^{\alpha\beta} = -\rho_0\left[\left(\frac{\partial\psi'}{\partial e_{\alpha\beta}}\right)_T\right]_\perp$$

as given in (2.13) (see also Maugin 1973b, p 254 and Carter and Quintana 1972, § 4).

‡ Equation $\mathscr{L}\psi' = (\mathscr{L}\epsilon' - T\mathscr{L}\eta') - \eta'\mathscr{L}T$ by the use of (2.25) can be written

$$\mathscr{L}\psi' = -\frac{1}{\rho}\theta^{\alpha\beta}\mathscr{L}e_{\alpha\beta} - \eta'\mathscr{L}T$$

on the one hand; and also as

$$\mathscr{L}\psi'(e_{\alpha\beta}, T) = \frac{\partial\psi'}{\partial e_{\alpha\beta}}\mathscr{L}e_{\alpha\beta} + \frac{\partial\psi'}{\partial T}\mathscr{L}T$$

on the other hand; recalling that $\mathscr{L}e_{\alpha\beta} = [\mathscr{L}e_{\alpha\beta}]_\perp$, (see (2.11)) and considering (2.4) the relation (2.13) follows by comparison of both expressions for $\mathscr{L}\psi'$.

as we neglected second- and higher-order terms in $e_{\alpha\beta}$. When $e_{\alpha\beta}=0$ and $T=T_0$, $\theta^{\alpha\beta}=[A^{\alpha\beta}]_\perp$ must be the residual stress in the initial reference state. Equation (2.18) (or (2.18*a*)) thus emerges as a generalised Hooke's law for a thermally coupled elastic solid (cf Bennoun 1965, pp 77 and 92, Barrabes 1975; see also Schmutzer 1968, p 425).

Because $[B^{\alpha\beta\gamma\delta}e_{\gamma\delta}]_\perp=[B^{\alpha\beta\gamma\delta}]_\perp[e_{\gamma\delta}]_\perp=[B^{\alpha\beta\gamma\delta}]_\perp e_{\gamma\delta}$ where the last step is possible due to (2.4), we can also deduce from (2.18) the equation for the rate of change of stress, bearing in mind (2.17):

$$-\mathscr{L}\theta^{\alpha\beta}=[B^{\alpha\beta\gamma\delta}]_\perp\mathscr{L}e_{\gamma\delta}+[C^{\alpha\beta}]_\perp\mathscr{L}T. \tag{2.19}$$

From the symmetry of $e_{\alpha\beta}(=e_{(\alpha\beta)})$, one can, by inspection, determine the symmetries of the coefficients in (2.16): $A^{\alpha\beta}=A^{(\alpha\beta)}$, $C^{\alpha\beta}=C^{(\alpha\beta)}$, $B^{\alpha\beta\gamma\delta}=B^{(\alpha\beta)(\gamma\delta)}=B^{(\gamma\delta)(\alpha\beta)}$. Therefore $[C^{\alpha\beta}]_\perp$ has six and $[B^{\alpha\beta\gamma\delta}]_\perp$ has twenty-one independent components. In an isotropic medium, in particular, the following relation is obtained:

$$[B^{\alpha\beta\gamma\delta}]_\perp=\lambda\overset{\perp}{g}{}^{\alpha\beta}\overset{\perp}{g}{}^{\gamma\delta}+2\mu\overset{\perp}{g}{}^{\alpha(\gamma}\overset{\perp}{g}{}^{\delta)\beta};\qquad [A^{\alpha\beta}]_\perp=A\overset{\perp}{g}{}^{\alpha\beta},\qquad [C^{\alpha\beta}]_\perp=-\beta\overset{\perp}{g}{}^{\alpha\beta}, \tag{2.20}$$

where λ and μ are Lamé's constants and β is proportional to the coefficient of thermal expansion. Using (2.20) equation (2.18) can be rewritten as:

$$-\theta^{\alpha\beta}=-A\overset{\perp}{g}{}^{\alpha\beta}+\lambda\overset{\perp}{g}{}^{\alpha\beta}[e^{\gamma}_{\gamma}]_\perp+2\mu[e^{\alpha\beta}]_\perp-\beta(T-T_0)\overset{\perp}{g}{}^{\alpha\beta}. \tag{2.18a}$$

The reversible specific entropy is, by the second equation of (2.13) and (2.15),

$$-\eta'=\left(\frac{\partial\psi'}{\partial T}\right)_e=\psi^{(1)}+\frac{1}{\rho_0}e_{\alpha\beta}C^{\alpha\beta}+(T-T_0)\psi^{(2)} \tag{2.21}$$

which, after being subjected to the $\mathscr{L}$ derivative and using (2.17) and (2.4), leaves:

$$-\mathscr{L}\eta'=\frac{1}{\rho_0}[C^{\alpha\beta}]_\perp\mathscr{L}e_{\alpha\beta}+\psi^{(2)}\mathscr{L}T. \tag{2.22}$$

Using the usual definition of specific heat c_e at constant strain,

$$c_e\equiv\left(\frac{\partial\epsilon}{\partial T}\right)_e=T\left(\frac{\partial\eta'}{\partial T}\right)_e \tag{2.23}$$

then $c_e=-T\psi^{(2)}$ by (2.21) and $[C^{\alpha\beta}]_\perp=-\partial\theta^{\alpha\beta}$ (by (2.18)); equation (2.22) then reads

$$\rho T\mathscr{L}\eta'=T\frac{\partial\theta^{\alpha\beta}}{\partial T}\mathscr{L}e_{\alpha\beta}+\rho c_e\mathscr{L}T, \tag{2.24}$$

and expressing its left-hand side by the use of Gibbs equation for reversible change:

$$\rho T\mathscr{L}\eta'=\rho\mathscr{L}\epsilon'+\theta^{\alpha\beta}\mathscr{L}e_{\alpha\beta} \tag{2.25}$$

we can finally write the equation

$$\rho\mathscr{L}\epsilon'=\rho c_e\mathscr{L}T+\left(T\frac{\partial\theta^{\alpha\beta}}{\partial T}-\theta^{\alpha\beta}\right)\mathscr{L}e_{\alpha\beta} \tag{2.26}$$

which will be useful to transform the term $\rho\mathscr{D}\epsilon$ in (2.8).

The tensor $\Pi^{\alpha\beta}$, representing the irreversible internal friction, which we will also write in the form

$$\Pi^{\alpha\beta} = \pi g^{\perp\alpha\beta} + \pi^{\alpha\beta}, \qquad \pi \equiv \tfrac{1}{3}\Pi^{\lambda}_{\lambda}, \qquad \pi^{\alpha\beta} \equiv \langle \Pi^{\alpha\beta} \rangle, \qquad \pi^{\alpha}_{\alpha} = 0 \tag{2.27}$$

is conventionally (in the frame of stationary non-equilibrium theory) connected with the rate of strain tensor by the proportionality relations

$$\pi = -\tfrac{1}{3}\lambda^{0} c\mathscr{L}e^{\alpha}_{\alpha}, \qquad \pi^{\alpha\beta} = -\bar{\lambda}\langle c\mathscr{L}e^{\alpha\beta}\rangle, \tag{2.28}$$

which in fluid theory are known as Newton's law and in viscoelasticity (together with (2.27) and (2.18)) are used to describe the so called Voigt solid (see I, equation (2.8)). Such constitutive equations describe the instantaneous dissipative stress response (π and $\pi^{\alpha\beta}$), arising from the rate of strain ($\mathscr{L}e^{\alpha}_{\alpha}$ and $\langle \mathscr{L}e^{\alpha\beta}\rangle$) which contradict physical causality and must therefore be corrected. A suitable correction is assured automatically if, instead of conventional stationary thermodynamics, the more exact non-stationary thermodynamics in the sense described by Müller (1966, 1967) is used.

In non-stationary thermodynamics, we do not alter the first principle, but we write the second principle as

$$\nabla_{\alpha} S^{\alpha} = \sigma \qquad \text{where} \qquad S^{\alpha} = cu^{\alpha}\rho\eta + \overset{\perp}{S}{}^{\alpha}$$

or

$$\rho c\mathscr{D}\eta + \nabla_{\alpha}\overset{\perp}{S}{}^{\alpha} = \sigma \qquad \text{and} \qquad \sigma \geqslant 0 \tag{2.29}$$

which is a statement of the entropy balance equation and of the Clausius–Duhem inequality, and where we have to retain explicitly all terms to order two (O(2)).

All quantities which, in the thermodynamical equilibrium state, do not vanish (e.g. ρ, T) are considered to be of O(0) and all quantities which vanish in thermal equilibrium (e.g. q^{α}, π, $\pi^{\alpha\beta}$) as well as the derivatives of all such quantities, are considered to be of O(1).

Such a requirement can be secured by two constitutive assumptions ((*a*) and (*b*)):

(*a*) The (specific) internal entropy η (as one of the three new quantities (η, $\overset{\perp}{S}{}^{\alpha}$, σ) appearing in the fully independent intrinsic second principle (2.29)) in a thermal non-equilibrium state depends explicitly, in addition to the usual variables (e.g. $e_{\alpha\beta}$, and T) for description of reversible processes, also on the dissipative fluxes characterising irreversible processes (i.e. q^{α}, π, $\pi^{\alpha\beta}$), as it must describe the irreversibility of non-equilibrium processes too. Therefore

$$\eta = \eta(e_{\alpha\beta}, T; q_{\alpha}, \pi_{\alpha\beta}, \pi). \tag{2.30}$$

In order to derive the appropriate generalised Gibbs equation we take the $\mathscr{L}$ derivative of (2.30) (using (2.4)):

$$\mathscr{L}\eta = \left[\frac{\partial\eta}{\partial e_{\alpha\beta}}\right]_{\perp}[\mathscr{L}e_{\alpha\beta}]_{\perp} + \frac{\partial\eta}{\partial T}\mathscr{L}T + \left[\frac{\partial\eta}{\partial q_{\alpha}}\right]_{\perp}[\mathscr{L}q_{\alpha}]_{\perp} + \left[\frac{\partial\eta}{\partial\pi_{\alpha\beta}}\right]_{\perp}[\mathscr{L}\pi_{\alpha\beta}]_{\perp} + \frac{\partial\eta}{\partial\pi}\mathscr{L}\pi. \tag{2.31}$$

We also require that the dependence of η and ϵ on the variables describing the reversible processes be the same as in reversible thermodynamics η', and ϵ', i.e.

$$\epsilon = \epsilon', \tag{2.32}$$

$$\left[\frac{\partial\eta}{\partial e_{\alpha\beta}}\right]_{\perp} = \left[\frac{\partial\eta'}{\partial e_{\alpha\beta}}\right]_{\perp} \equiv \frac{1}{T}\left(\frac{1}{\rho}\theta^{\alpha\beta} + \left[\frac{\partial\epsilon'}{\partial e_{\alpha\beta}}\right]_{\perp}\right), \tag{2.33}$$

$$\frac{\partial\eta}{\partial T}=\frac{\partial\eta'}{\partial T}\equiv\frac{1}{T}\frac{\partial\epsilon'}{\partial T}. \tag{2.34}$$

Being concerned with an isotropic medium, where the dissipation can be considered as a random-walk process, the η may be alternatively expressed as a function of invariants of dissipative fluxes, so we can also write, instead of (2.30):

$$\eta=\eta(e_{\alpha\beta},\,T;\,q^\alpha q_\alpha,\,\pi^{\alpha\beta}\pi_{\alpha\beta},\,\pi). \tag{2.35}$$

Expanding η in a Taylor series about the reference state $\eta(e_{\alpha\beta}, T; 0, 0, 0)\equiv\eta_*=\eta'$ to order two in the dissipative fluxes, namely

$$\begin{aligned}\eta(e_{\alpha\beta},\,T;\,q^\alpha q_\alpha,\,\pi^{\alpha\beta}\pi_{\alpha\beta},\,\pi)-\eta'&\\ =-\frac{1}{T}(\bar\vartheta(e_{\alpha\beta},\,T)\pi+\tfrac{1}{2}\bar\alpha(e_{\alpha\beta},\,T)q^\alpha q_\alpha+\tfrac{1}{2}\bar\beta(e_{\alpha\beta},\,T)\pi^{\alpha\beta}\pi_{\alpha\beta}+\tfrac{1}{2}\bar\gamma(e_{\alpha\beta},\,T)\pi^2)&\\ \equiv\Delta\eta&\end{aligned} \tag{2.36}$$

(where we have denoted $-(1/T)\bar\vartheta(e_{\alpha\beta}, T)=(\partial\eta/\partial\pi)_*$ etc) we can find

$$\left[\frac{\partial\eta}{\partial q_\alpha}\right]_\perp=-\frac{1}{T}\bar\alpha\overset{\perp}{q}{}^\alpha,\qquad\left[\frac{\partial\eta}{\partial\pi_{\alpha\beta}}\right]_\perp=-\frac{1}{T}\bar\beta[\pi^{\alpha\beta}]_\perp,\qquad\frac{\partial\eta}{\partial\pi}=-\frac{1}{T}\bar\gamma\pi. \tag{2.37}$$

As the entropy of the insulated system can only increase, i.e.

$$\eta\leqslant\eta'(e_{\alpha\beta},\,T)\qquad\text{or}\qquad\Delta\eta\equiv-\frac{1}{2T}(2\bar\vartheta\pi+\bar\alpha q^\alpha q_\alpha+\bar\beta\pi^{\alpha\beta}\pi_{\alpha\beta}+\bar\gamma\pi^2)\leqslant 0 \tag{2.38}$$

for arbitrary combinations of q^α, $\pi^{\alpha\beta}$ and π we may conclude that

$$\bar\vartheta(e_{\alpha\beta},\,T)=0,\qquad\bar\alpha(e_{\alpha\beta},\,T)\geqslant 0,\qquad\bar\beta(e_{\alpha\beta},\,T)\geqslant 0,\qquad\bar\gamma(e_{\alpha\beta},\,T)\geqslant 0. \tag{2.39}$$

Making use of relations (2.32)–(2.34), (2.37) and the first equation of (2.39), equation (2.31) representing Gibbs equation for non-stationary processes reads:

$$\rho T\mathscr{L}\eta=\rho\mathscr{L}\epsilon+\theta^{\alpha\beta}\mathscr{L}e_{\alpha\beta}-\rho\bar\alpha q^\alpha[\mathscr{L}q_\alpha]_\perp-\rho\bar\beta\pi^{\alpha\beta}[\mathscr{L}\pi_{\alpha\beta}]_\perp-\rho\bar\gamma\pi\mathscr{L}\pi. \tag{2.40}$$

One notices that the two first terms on the right-hand side (the only ones existing in reversible and also stationary irreversible thermodynamics) are of O(1) (assuming that $A^{\alpha\beta}\neq 0$ in (2.18)) while the remaining terms which are new are of O(2).

(*b*) The conductive part of the entropy flux $\overset{\perp}{S}{}^\alpha$ (see (2.29)), according to the conventional irreversible thermodynamics, is proportional to only one dissipation flux, i.e. heat flux, which causes $\overset{\perp}{S}{}^\alpha$ to be of O(1). Only if terms of O(2) are included is one able to combine all the dissipation fluxes into a vector of entropy flux. Thus a generalised definition of $\overset{\perp}{S}{}^\alpha$ may be given as

$$\overset{\perp}{S}{}^\alpha\equiv\frac{1}{T}(q^\alpha-N\pi q^\alpha-M\pi^{\alpha\beta}q_\beta) \tag{2.41}$$

where the scalar coefficients N and M may be dependent on $e_{\alpha\beta}$ and T.

The five new coefficients $\bar\alpha$, $\bar\beta$, $\bar\gamma$, N and M characterising the state of the medium have to be in conformity with the phenomenological transport equations which lead to the non-negative entropy production $\sigma\geqslant 0$ (see (2.29)). In other words the transport equations for dissipative fluxes can be deduced from the preceding equations, by the supposition that σ is a quadratic form of dissipative fluxes or a linear form of non-trivial invariants of dissipative fluxes. So, upon combining the third equation of (2.29), equations (2.40), (2.41) and (2.8) (together with (2.12) and (2.27)) and (2.11) retaining only the terms to O(2), and using the relations

$$\pi^{\alpha\beta}[\mathscr{L}e_{\alpha\beta}]_\perp = \pi^{\alpha\beta}\langle[\mathscr{L}e_{\alpha\beta}]_\perp\rangle, \qquad \pi^{\alpha\beta}[\nabla_\alpha q_\beta]_\perp = \pi^{\alpha\beta}\langle[\nabla_{(\alpha}q_{\beta)}]_\perp\rangle,$$
$$\pi^{\alpha\beta}[\mathscr{L}\pi_{\alpha\beta}]_\perp = \pi^{\alpha\beta}\langle[\mathscr{L}\pi_{\alpha\beta}]_\perp\rangle, \tag{2.42}$$

which follow by (2.5) and (2.4) one obtains:

$$\sigma T = -\pi([c\mathscr{L}e^\gamma_\gamma]_\perp + \rho\bar{\gamma}c\mathscr{L}\pi + N\nabla_\alpha q^\alpha)$$
$$-\pi^{\alpha\beta}(\langle[c\mathscr{L}e_{\alpha\beta}]_\perp\rangle + \rho\bar{\beta}\langle[c\mathscr{L}\pi_{\alpha\beta}]_\perp\rangle + M\langle[\nabla_{(\alpha}q_{\beta)}]_\perp\rangle)$$
$$-q^\alpha\left(\frac{1}{T}(\nabla_\alpha T + T\mathscr{D}u_\alpha) + \rho\bar{\alpha}[c\mathscr{L}q_\alpha]_\perp + N\nabla_\alpha\pi + M\nabla^\sigma\pi_{\sigma\alpha}\right). \tag{2.43}$$

The requirement that the scalar σT produce a quadratic form in π, $\pi^{\alpha\beta}$, and q^α leads to the following assumption for the form of transport equation (writing $c\mathscr{L}\pi_{\alpha\beta} = \dot{\pi}_{\alpha\beta}$ etc):

$$\pi + \tau^0\dot{\pi} + \tfrac{1}{3}\lambda^0(\dot{e}^\sigma_\sigma + N\nabla_\sigma q^\sigma) = 0 \tag{2.44}$$

$$\pi_{\alpha\beta} + \bar{\tau}[\dot{\pi}_{\alpha\beta}]_\perp + \bar{\lambda}(\langle[\dot{e}_{\alpha\beta}]_\perp\rangle + M\langle[\nabla_{(\alpha}q_{\beta)}]_\perp\rangle) = 0 \tag{2.45}$$

$$q_\alpha + \tau[\dot{q}_\alpha]_\perp + \kappa T\left(\frac{[\nabla_\alpha T + T\mathscr{D}u_\alpha]_\perp}{T} + N[\nabla_\alpha\pi]_\perp + M[\nabla_\sigma\pi^\sigma_\alpha]_\perp\right) = 0 \tag{2.46}$$

where the coefficients of proportionality λ^0, $\bar{\lambda}$ and κ are the so called transport coefficients of bulk and shear viscosity and of thermal conductivity. We introduced also the notation

$$\tau^0 \equiv \tfrac{1}{3}\lambda^0\rho\bar{\gamma}, \qquad \bar{\tau} \equiv \bar{\lambda}\rho\bar{\beta}, \qquad \tau \equiv \kappa T\rho\bar{\alpha}, \tag{2.47}$$

for the so called relaxation coefficients. With the equations (2.44)–(2.46) and (2.43) the Gibbs–Duhem inequality takes the desired form

$$\sigma \equiv \frac{1}{T}\left(\frac{3}{\lambda^0}\pi^2 + \frac{1}{\bar{\lambda}}\pi^{\alpha\beta}\pi_{\alpha\beta} + \frac{1}{\kappa T}q^\alpha q_\alpha\right) \geqslant 0. \tag{2.48}$$

The transport equations (2.44)–(2.46) contain five new coefficients (in general dependent on $e_{\alpha\beta}$ and T) as compared to the conventional stationary theory; these are the three relaxation times (2.47) and the two cross-effect coupling coefficients N (heat–bulk viscosity) and M (heat–shear viscosity). The forms of the non-stationary transport equations (2.44)–(2.46), the Gibbs equation (2.40) and the definition (2.41) are in agreement with the corresponding equations deduced from relativistic kinetic theory (13- and 14-moment approach; see Kranyš 1972, 1976). The transport equations with $N = M = 0$, i.e. completed only by the relaxation terms, were proposed also by Kranyš (1966a, b, 1967) in order to ensure relativistic causality of the conventional parabolic theory. Maugin (1973a, 1974b) also studied these transport equations in order to show that relaxation terms conform with the recent phenomenological axiomatisation of constitutive theory. However, we do not agree with all his conclusions. For example, he takes issue with this author that the heat transport equation analogous to our equation (2.46) is not 'objective' or 'rheologically invariant' by which he means that for the relaxation term the form $\tau c\mathscr{D}q_\alpha$ was used instead of the correct form, $\tau[c\mathscr{L}q_\alpha]_\perp$ which holds if $q_\alpha = \overset{\perp}{q}_\alpha$. As far as the projection

$[\ ]_\perp$ is concerned, in Kranyš (1966a) the heat flux is written as $q_\alpha = u_\alpha \overset{\parallel}{q} + \overset{\perp}{q}_\alpha$, $(\overset{\parallel}{q} \neq 0)$ so the criticism does not apply and in Kranyš (1966b, 1967), in the main text, this omission does exist but was amended at the end of the second article. Secondly, replacement of $\mathscr{L}q_\alpha$ by $\mathscr{D}q_\alpha$ is correct in the theory linearised with respect to dissipative fluxes which is our concern here. In this case (or if u_α = constant) $[\mathscr{L}q_\alpha]_\perp = \mathscr{D}q_\alpha$ holds (see equation (3.6)). Further Maugin was not able, by his method, to confirm, for a fluid, the existence of a term claimed by Stewart which is analogous to our term $\kappa TM[\nabla_\sigma \pi^\sigma_\alpha]_\perp$, whose existence is conclusively confirmed by the kinetic theory. It may appear perhaps surprising that we so often mention kinetic theory in talking of an elastic solid. As was pointed out in I, the thermal energy in a solid is transported by free electrons and phonons. Both are usually described as a gas of quasi-particles satisfying the Boltzmann equation, so we have to expect that the appropriate transport equations for these quasi-particles will be very similar to those obtained from the kinetic theory of a gas of molecules.

The transport equations (2.44)–(2.46), in the non-relativistic limit go over into the corresponding classical equations (2.9)–(2.11) given in I†. Only one small term is without analogy with a non-relativistic description: $\kappa T[c\mathscr{D}cu_\alpha]_\perp/c^2$ is a part of Eckart's relativistic temperature gradient $[\nabla_\alpha T + T\mathscr{D}u_\alpha]_\perp$ which has sometimes been unjustly criticised (e.g. Bennoun 1965, p 68) but which is presently considered to be quite natural (see Maugin 1974c).

The resulting form of the stress–strain relation for our dissipative solid follows by combining equations (2.18*a*), (2.44) and (2.45):

$$\begin{aligned}\theta_{\alpha\beta} + \Pi_{\alpha\beta} &+ (\tau^0 - \bar{\tau})\tfrac{1}{3}\overset{\perp}{g}_{\alpha\beta}[\dot{\Pi}^\sigma_\sigma]_\perp + \bar{\tau}[\dot{\Pi}_{\alpha\beta}]_\perp \\ &= -[3\lambda\tfrac{1}{3}\overset{\perp}{g}_{\alpha\beta}[e^\sigma_\sigma]_\perp + 2\mu[e_{\alpha\beta}]_\perp - A\overset{\perp}{g}_{\alpha\beta} - (T - T_0)\beta\overset{\perp}{g}_{\alpha\beta} + 3\lambda'\tfrac{1}{3}\overset{\perp}{g}_{\alpha\beta}[\dot{e}^\sigma_\sigma]_\perp \\ &\quad + 2\mu'[\dot{e}_{\alpha\beta}]_\perp + (\lambda^0 N - \bar{\lambda}M)\tfrac{1}{3}\overset{\perp}{g}_{\alpha\beta}[\nabla_\gamma q^\gamma]_\perp + \bar{\lambda}M[\nabla_{(\alpha}q_{\beta)}]_\perp] \end{aligned} \tag{2.49}$$

where also (2.12) and (2.27)‡ have been used. Equation (2.49) represents our generalised form of the idealised Maxwell–Voigt–Meyer equation of viscoelasticity

$$w_{\alpha\beta} + \hat{\tau}[\dot{w}_{\alpha\beta}]_\perp = \hat{A}_{\alpha\beta}{}^{\gamma\delta}[e_{\gamma\delta}]_\perp + \hat{B}_{\alpha\beta}{}^{\gamma\delta}[\dot{e}_{\gamma\delta}]_\perp \tag{2.50}$$

which is called the Maxwell equation if $\hat{A}_{\alpha\beta}{}^{\gamma\delta} = 0$ and the Meyer–Kelvin–Voigt equation if $\hat{\tau} = 0$. In relativity, an equation of type (2.50) was studied by Maugin (1973a, 1974a) with $\hat{\tau} = 0$. For an equation of type (2.50), where the total stress $w_{\alpha\beta}$ is relaxed, a thermodynamics has not yet been constructed (Eringen 1967, p 330). We notice that in (2.49) only the dissipative fluxes $\Pi_{\alpha\beta}$ are *relaxed*, not the reversible quantities like $\theta_{\alpha\beta}$ as the rôle of 'driving force' and its 'response' must not be interchanged since the relaxation terms, containing time derivatives, are not invariant under the substitution $t \to -t$. One could also interpret equation (2.49) as a generalised Hooke's law for a dissipative elastic medium.

The first law of thermodynamics (2.8) can be written as follows, if use is made of (2.12), (2.26), (2.32), the relation (2.18*a*) and (2.11)

$$\rho c_e c\mathscr{L}T + \beta Tc\mathscr{L}e^\sigma_\sigma + \Pi^{\alpha\beta}c\mathscr{L}e_{\alpha\beta} + (\nabla_\sigma q^\sigma + q^\sigma\mathscr{D}u_\sigma) = 0. \tag{2.51}$$

† Due to slightly different notation in I we must make the substitutions $\tau_{kl} \to \pi_{kl}$, $\pi \to 3\pi$, $N \to \frac{1}{3}N$ and $\mathscr{N} \to \frac{1}{2}\mathscr{N}$.
‡ $\lambda^0 = 3\lambda' + 2\mu'$, $\bar{\lambda} = 2\mu'$ where λ' and μ' are moduli of viscosity which correspond to Lamé's constants, were also used.

The reversible part of the pressure tensor $\theta^{\alpha\beta}$ can be written

$$\theta^{\alpha\beta}=p\overset{\perp}{g}{}^{\alpha\beta}+\bar{\theta}^{\alpha\beta};\qquad \bar{\theta}^{\alpha}_{\alpha}=0,\qquad (p=\tfrac{1}{3}\theta^{\alpha}_{\alpha}), \tag{2.52}$$

where p is the scalar isotropic pressure and $\bar{\theta}^{\alpha\beta}$ is the pressure deviator (shear tensor). For example, in the case of a Hookean thermoelastic solid (see (2.18a)) we have

$$\begin{aligned} p&=A+\beta(T-T_0)-(\lambda+\tfrac{2}{3}\mu)e^{\gamma}_{\gamma};\\ \bar{\theta}^{\alpha\beta}&=-2\mu([e^{\alpha\beta}]_{\perp}-\tfrac{1}{3}\overset{\perp}{g}{}^{\alpha\beta}[e^{\gamma}_{\gamma}]_{\perp})=-2\mu\langle[e^{\alpha\beta}]_{\perp}\rangle. \end{aligned} \tag{2.53}$$

The case of $p\neq 0$ and $\bar{\theta}^{\alpha\beta}=0$, characterises *fluid continua* (called compressible if $\partial p/\partial e^{\gamma}_{\gamma}\neq 0$), which cannot resist shear tension. The case $\bar{\theta}^{\alpha\beta}\neq 0$ and $p=0$ is an example of purely *non-fluid continua* (solids) characterised by a rigidity modulus μ which are able to withstand only a shear tension but are incompressible. In general non-fluid continua (solids) (which are treated in this paper) are both compressible and shear-resistant at the same time.

The decomposition (2.52) can help us to determine, for example, the contribution of each part of $\theta^{\alpha\beta}$ in the governing equations. The equation of motion (2.7) by virtue of the first of the equations (2.12), and also (2.52) reads

$$\begin{aligned}(\rho\epsilon+p)\mathscr{D}u^{\sigma}&+(\overset{\perp}{\nabla}{}^{\sigma}p+\nabla_{\alpha}\bar{\theta}^{\alpha\sigma}+\nabla_{\alpha}\Pi^{\alpha\sigma})\\ &-u^{\sigma}(\bar{\theta}^{\alpha\beta}+\Pi^{\alpha\beta})\overset{\perp}{\nabla}_{(\alpha}u_{\beta)}+\frac{1}{c}(\mathscr{D}q^{\sigma}-u^{\sigma}q^{\beta}\mathscr{D}u_{\beta})=0,\end{aligned} \tag{2.54}$$

where the relation $p\nabla_{\alpha}\overset{\perp}{g}{}^{\alpha\sigma}=p(u^{\sigma}\nabla_{\alpha}u^{\alpha}+\mathscr{D}u^{\sigma})$ has been used. We notice that the coefficient of acceleration $c\mathscr{D}cu^{\sigma}$ contains, beside ρ (as in Newtonian mechanics), a typical relativistic factor $[\epsilon+(p/\rho)]/c^2$ which is well known from the relativistic dynamics of fluid continua and which also appears, of course, in the general case of non-fluid continua.

Upon application of the decomposition (2.52) in the Gibbs equation (2.40) (or in (2.25)) we see that our new term

$$\theta^{\alpha\beta}\mathscr{L}e_{\alpha\beta}=\rho p\mathscr{L}\left(\frac{1}{\rho}\right)+\bar{\theta}^{\alpha\beta}\mathscr{L}e_{\alpha\beta} \tag{2.55}$$

by (2.11) and (2.6) splits into the familiar term characteristic of a fluid (the only one included in the original Müller theory) and a second term $\bar{\theta}^{\alpha\beta}\mathscr{L}e_{\alpha\beta}$ which represents the net contribution from the purely non-fluid (i.e. rigidity) properties of continua.

3. Review of the governing equations of the theory

We noticed that for deriving (2.43) and the transport equations (involving practically only terms of O(1)), the second and first principles, both expressed by non-linear equations containing terms of O(2) were needed. The governing equations of our theory are formed now by the conservation equations (2.6), (2.7) (with (2.12), (2.18) and (2.27)), (2.51) and by the transport equations (2.44), (2.45) and (2.46). By virtue of (2.11) and the supplementary conditions given by the second equation of (2.1), (2.2) and the last equation of (2.27), and $\pi_{\alpha\beta}u^{\beta}=0$, we have, in the special relativity case, 14 equations (not including the supplementary conditions) for determining the 14 unknown quantities:

$$\rho, \quad u_\alpha, \quad T, \quad \pi, \quad \pi_{\alpha\beta}, \quad q_\alpha; \tag{3.1}$$

which form the determinate system, without recourse to the notions involved in the second principle. The entropy balance equation and Gibbs–Duhem inequality, are used only if some further, purely thermodynamical considerations, are called for. In the general relativity case, we have, besides the equations mentioned above, the Einstein field equations:

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -\chi T_{\alpha\beta} \tag{3.2}$$

and, besides the 14 unknowns (3.1), 10 additional unknown components of the metrical tensor $g_{\alpha\beta}$.

If one intended to use the non-linear stress–strain relation (finite strain elasticity) it would be consistent to keep the conservation equations (2.6), (2.7) and (2.51) non-linear, i.e. involving also terms of O(2), while the transport equations (2.44)–(2.46) involving terms of O(1) remain linear at least with respect to the dissipative fluxes.

Consistently with our assumption of infinitesimal strain leading to the linear stress–strain relation we wish now, for the sake of simplicity, to retain only the terms of O(1) in all conservation equations, which corresponds, to all intents and purposes, to the linearisation of these equations. All coefficients of the differential equations will be considered as having constant values corresponding to the reference state, i.e. in the coefficients we set

$$T \to T_0 = T_{\text{eq}}, \quad \rho \to \rho_0, \quad \epsilon \to \epsilon_0, \quad p \to p_0 = A,$$

but (3.1*a*)

$$e^{\alpha\beta} \to 0, \quad \ldots, \quad \pi^{\alpha\beta} \to 0, \quad q^\alpha \to 0, \quad \pi \to 0$$

and we will drop the suffix eq or 0.

Before writing down the governing equations it is necessary to treat the equation of motion. Equation (2.18*a*) by virtue of (2.11) can be written as

$$\mathscr{L}\theta^{\alpha\beta} = -2\mu s^{\alpha\beta} + (-\lambda s^\gamma_\gamma + \beta \mathscr{L} T)\overset{\perp}{g}{}^{\alpha\beta} \tag{3.3}$$

which, after differentiating, leaves:

$$\nabla_\alpha \mathscr{L}\theta^{\alpha\beta} - \beta \overset{\perp}{\nabla}{}^\beta \mathscr{L} T = -2\mu \nabla_\alpha s^{\alpha\beta} - \lambda \overset{\perp}{\nabla}{}^\beta s^\gamma_\gamma = -\mu \overset{\perp}{\nabla}_\alpha \overset{\perp}{\nabla}{}^\alpha u^\beta - (\mu+\lambda) \overset{\perp}{\nabla}{}^\beta \overset{\perp}{\nabla}_\gamma u^\gamma \tag{3.4}$$

where the term $(-\lambda s^\gamma_\gamma + \beta \mathscr{L} T)\nabla_\alpha \overset{\perp}{g}{}^{\alpha\beta}$ was neglected as being of O(2), and the first equation of (2.5) and (2.10) were used. Applying the $c\mathscr{L}$ derivative on equation (2.7) (or (2.54)), where the free indices were lowered, dropping terms of O(2) and using (3.4), the second equation of (A.2) and symbols $c\mathscr{L}\pi = \dot{\pi}$ etc, we obtain the desired equation of motion

$$\frac{(p+\rho\epsilon)}{c^2} c\ddot{u}_\sigma - \mu \overset{\perp}{\nabla}_\alpha \overset{\perp}{\nabla}{}^\alpha c u_\sigma - (\mu+\lambda) \overset{\perp}{\nabla}_\sigma \overset{\perp}{\nabla}{}^\gamma c u_\gamma + \beta \overset{\perp}{\nabla}_\sigma \dot{T} + \nabla^\alpha \dot{\pi}_{\alpha\sigma} + \overset{\perp}{\nabla}_\sigma \dot{\pi} + \frac{1}{c^2} \ddot{q}_\sigma = 0. \tag{3.5}$$

In order to perform a linearisation of the governing equations for special relativity theory, one needs to realise that the $\mathscr{L}$ derivative of all tensors of O(1) reduces itself to the $\mathscr{D} \equiv u^\alpha \partial_\alpha$ derivative as is evident from the formulae in appendix 1, and $\nabla^\alpha \to \partial^\alpha$.

Also, for example

$$\frac{1}{c}[q_\alpha]_\perp \equiv g^\sigma_\alpha \mathscr{D} q_\sigma = \mathscr{D} q_\alpha + u_\alpha u^\sigma \mathscr{D} q_\sigma = \mathscr{D} q_\alpha - u_\alpha q_\sigma \mathscr{D} u^\sigma \approx \mathscr{D} q_\alpha \tag{3.6}$$

where the first equation of (2.2) was used and the term $q_\sigma \mathscr{D} u^\sigma$ was neglected as it is of O(2). Similarly one deduces, also,

$$[\dot{\pi}_{\alpha\beta}]_\perp = c\mathscr{D}\pi_{\alpha\beta}, \qquad [\nabla_\alpha \pi]_\perp = \overset{\perp}{\partial}_\alpha \pi, \qquad \langle [\dot{e}_{\alpha\beta}]_\perp \rangle = \langle \overset{\perp}{\partial}_{(\alpha} u_{\beta)} \rangle, \qquad \ldots \tag{3.7}$$

Keeping this in mind we can write the 14 governing equations of our theory (i.e. (2.6), (3.5), (2.51), (2.44), (2.45) and (2.46)) in linearised form with differentiated supplementary conditions:

$$c\mathscr{D}\rho + \rho\,\partial_\alpha c u^\alpha = 0, \tag{3.8}$$

$$\frac{(p+\rho\epsilon)}{c^2} c^2 \mathscr{D}^2 c u^\beta - (\mu+\lambda)\overset{\perp}{\partial}{}^\beta \overset{\perp}{\partial}_\gamma c u^\gamma - \mu \overset{\perp}{\partial}_\alpha \overset{\perp}{\partial}{}^\alpha c u^\beta + \beta c \mathscr{D} \overset{\perp}{\partial}{}^\beta T + c\mathscr{D}\overset{\perp}{\partial}_\alpha \pi^{\alpha\beta}$$

$$+ c\mathscr{D}\overset{\perp}{\partial}{}^\beta \pi + \frac{1}{c^2} c^2 \mathscr{D}^2 q^\beta = 0, \tag{3.9}$$

$$\rho c_\epsilon c \mathscr{D} T + \beta T c\, \partial^\alpha u_\alpha + \partial_\alpha q^\alpha = 0, \tag{3.10}$$

$$\pi + \tau^0 c\mathscr{D}\pi + \tfrac{1}{3}\lambda^0 (\overset{\perp}{\partial}{}^\alpha c u_\alpha + N \partial_\lambda q^\lambda) = 0, \tag{3.11}$$

$$\pi^{\alpha\beta} + \bar{\tau} c \mathscr{D} \pi^{\alpha\beta} + \bar{\lambda}(\langle \overset{\perp}{\partial}{}^{(\alpha} u^{\beta)} \rangle + M \langle \partial^{(\alpha} q^{\beta)} \rangle) = 0, \tag{3.12}$$

$$q^\alpha + \tau c \mathscr{D} q^\alpha + \kappa\{[\overset{\perp}{\partial}{}^\alpha T + (T/c^2) c \mathscr{D} c u^\alpha] + NT\overset{\perp}{\partial}{}^\alpha \pi + MT \overset{\perp}{\partial}_\sigma \pi^{\alpha\sigma}\} = 0, \tag{3.13}$$

$$u_\alpha \mathscr{D} u^\alpha = 0, \qquad u_\alpha \mathscr{D} q^\alpha = 0, \qquad u_\alpha \mathscr{D} \pi^{\alpha\beta} = 0, \qquad \pi^\alpha_\alpha = 0. \tag{3.14}$$

4. The 14 linearised governing equations and their Fourier transforms

We will apply the present theory to a study of propagation modes (in the frame of special relativity theory) in an unbounded space filled with an immobile isotropic dissipative elastic medium in thermodynamical equilibrium and in mechanical equilibrium, and in which there is a *forced disturbance* of very small amplitude. Therefore, we assume that, for this problem, the governing equations (3.8)–(3.14), linearised near this equilibrium reference state, can be used.

In seeking a solution to the system of 14 linear partial differential equations (3.8)–(3.14), we assume each of the unknown functions (3.1) to have the form of a propagating *plane wave*:

$$Q - Q_{eq} = \hat{Q}\, e^{-iK^\sigma x_\sigma}. \tag{4.1}$$

This corresponds to a Fourier transform in time and space. K^σ is the four-wavevector, in the local rest frame $K^\sigma = (\omega/c, \boldsymbol{k})$, and the phase velocity is defined as $W = \omega/|\boldsymbol{k}|$. The relation $K^\sigma K_\sigma \geqslant 0$ holds, i.e. K^σ is a space-like or null vector which is the necessary condition for having $W \leqslant c$. An invariant decomposition of the four-wavevector into longitudinal and transverse parts with respect to the world line of the appropriate mass element is

$$K^\alpha = -u^\alpha \overset{\parallel}{K} + \overset{\perp}{K}{}^\alpha, \qquad \overset{\perp}{K}{}^\alpha = \overset{\perp}{n}{}^\alpha \overset{\perp}{K}, \qquad \overset{\perp}{n}{}^\alpha \overset{\perp}{n}_\alpha = +1. \tag{4.2}$$

Then the frequency and three-wavevector can be expressed in the invariant form

$$\omega = c\overset{\parallel}{K} = cK^{\alpha}u_{\alpha}, \qquad |\boldsymbol{k}| = \overset{\perp}{K} \equiv \overset{\perp}{n}_{\alpha}\overset{\perp}{g}{}^{\alpha}_{\beta}K^{\beta} \qquad \text{and} \qquad W = \frac{c\overset{\parallel}{K}}{\overset{\perp}{K}} = \frac{\omega}{\overset{\perp}{K}}. \tag{4.3}$$

In order to study the polarisation of waves in a simple manner, we choose a special local rest frame with x^3 pointing in the direction of propagation of the plane wave, i.e. $\overset{\perp}{n}{}^{\alpha} = (0; 0, 0, 1)$. As we wish to investigate *forced* sound waves (a description of sound propagation arising, e.g., from an oscillating piston), ω will always be real while the wavevector remains complex. The real part of the refractive index of the wave $\mathcal{N}^* = c/W = c\overset{\perp}{K}/\omega$ corresponds to a propagation phenomenon and $\mathrm{Im}(\mathcal{N}^*)$ to the attenuation.

Inserting into the set of our equations (3.8)–(3.14) for each of the unknown functions (3.1) a plane wave solution (4.1), one obtains (as $\partial^{\beta}Q \to -\mathrm{i}K^{\beta}\hat{Q}$, $c\mathscr{D}Q \to -\mathrm{i}\omega\hat{Q}$, $\overset{\perp}{\partial}{}^{\beta}Q \to -\mathrm{i}\overset{\perp}{K}{}^{\beta}\hat{Q}$):

$$\omega\hat{\rho} + \rho K_{\alpha}c\hat{u}^{\alpha} = 0, \tag{4.4}$$

$$\frac{(p+\rho\epsilon)}{c^2}\omega^2 c\hat{u}^{\beta} - (\mu+\lambda)\overset{\perp}{K}{}^{\beta}\overset{\perp}{K}_{\gamma}c\hat{u}^{\gamma} - \mu\overset{\perp}{K}_{\alpha}\overset{\perp}{K}{}^{\alpha}c\hat{u}^{\beta} + \beta\omega\overset{\perp}{K}{}^{\beta}\hat{T} + \omega\overset{\perp}{K}_{\alpha}\hat{\pi}^{\alpha\beta}$$

$$+\omega\overset{\perp}{K}{}^{\beta}\hat{\pi} + \frac{\omega^2}{c^2}\hat{q}^{\beta} = 0 \tag{4.5}$$

$$\rho c_{\epsilon}\omega\hat{T} + \beta TcK^{\alpha}\hat{u}_{\alpha} + K_{\alpha}\hat{q}^{\alpha} = 0, \tag{4.6}$$

$$\mathrm{i}\hat{\pi} + \tau^0\omega\hat{\pi} + \tfrac{1}{3}\lambda^0(\overset{\perp}{K}{}^{\alpha}c\hat{u}_{\alpha} + NK_{\lambda}\hat{q}^{\lambda}) = 0, \tag{4.7}$$

$$\mathrm{i}\hat{\pi}^{\alpha\beta} + \bar{\tau}\omega\hat{\pi}^{\alpha\beta} + \bar{\lambda}(\langle\overset{\perp}{K}{}^{(\alpha}c\hat{u}^{\beta)}\rangle + M\langle K^{(\alpha}\hat{g}^{\beta)}\rangle) = 0, \tag{4.8}$$

$$\mathrm{i}\hat{q}^{\alpha} + \tau\omega\hat{q}^{\alpha} + \kappa\left[\left(K^{\alpha}\hat{T} + \frac{T}{c^2}\omega c\hat{u}^{\alpha}\right) + NT\overset{\perp}{K}{}^{\alpha}\hat{\pi} + MT\overset{\perp}{K}_{\sigma}\hat{\pi}^{\sigma\alpha}\right] = 0, \tag{4.9}$$

$$u_{\alpha}\hat{u}^{\alpha} = 0, \qquad u_{\alpha}\hat{q}^{\alpha} = 0, \qquad u_{\alpha}\hat{\pi}^{\alpha\beta} = 0, \qquad \hat{\pi}^{\alpha}_{\alpha} = 0. \tag{4.10}$$

The equations (4.4)–(4.9) form a set of twenty-one homogeneous equations in the unknowns (3.1) of which only fourteen are independent because of (4.10). With $\overset{\perp}{n}{}^{\alpha} = (0; 0, 0, 1)$, i.e. $\overset{\perp}{K}{}^{\alpha} = (0; 0, 0, \overset{\perp}{K})$ and $u^{\alpha} = (1; 0, 0, 0)$ (immobile medium), $g_{\alpha\beta} = (-1; 1, 1, 1)$, $\overset{\perp}{g}_{\alpha\beta} = (0; 1, 1, 1)$, conditions (4.10) require

$$\hat{u}^0 = 0, \qquad \hat{q}^0 = 0, \qquad \hat{\pi}^{0\alpha} = 0. \tag{4.11}$$

Due to the constraint $\hat{\pi}^{\alpha}_{\alpha} = 0$ of (4.10), $\pi^{11} = -\pi^{22} - \pi^{33}$ is not an independent quantity and will not be considered in our system of equations (see also Kranyš 1976, 1972 for details about the same technique).

Taking all these formulae into account, equations (4.4)–(4.9) together with:

$$\bar{N} = \left(1 + \frac{\mathrm{i}}{\tau^0\omega}\right), \qquad \bar{B} = \left(1 + \frac{\mathrm{i}}{\bar{\tau}\omega}\right), \qquad \bar{Z} = \left(1 + \frac{\mathrm{i}}{\tau\omega}\right), \tag{4.12}$$

can be put in the matrix form (4.13).

$$
\left(\begin{array}{cccc|cccc|cccccc}
\frac{c^2}{\overset{*}{c}{}_{\perp}^{2}}\overset{*}{W}{}^2-1 & \cdot & \overset{*}{W} & \overset{*}{W}{}^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \overset{*}{W}\bar{B} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{1}{2}\bar{\Lambda}_{\perp}\frac{c^2}{\epsilon} & \cdot & \overset{*}{W}\bar{B} & \frac{1}{2}\bar{\Lambda}_{\perp}\mathcal{M}_{\perp}\frac{c_{\perp}}{v} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Lambda_{\perp}\frac{v^2}{\epsilon}\overset{*}{W} & \cdot & \Lambda_{\perp}\mathcal{M}_{\perp}\frac{vc_{\perp}}{c^2} & \overset{*}{W}\bar{Z} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \frac{c^2}{\overset{*}{c}{}_{\perp}^{2}}\overset{*}{W}{}^2-1 & \cdot & \overset{*}{W} & \overset{*}{W}{}^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \overset{*}{W}\bar{B} & \cdot & \cdot & \cdot & -\frac{1}{3}\bar{\Lambda}_{\perp}\frac{c^2}{\epsilon} & \cdot & \cdot & \cdot & -\frac{1}{3}\bar{\Lambda}\mathcal{M}_{\perp}\frac{c_{\perp}}{v} \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{2}\bar{\Lambda}_{\perp}\frac{c^2}{\epsilon} & \cdot & \overset{*}{W}\bar{B} & \frac{1}{2}\bar{\Lambda}_{\perp}\mathcal{M}_{\perp}\frac{c_{\perp}}{v} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \Lambda_{\perp}\frac{v^2}{\epsilon}\overset{*}{W} & \cdot & \Lambda_{\perp}\mathcal{M}_{\perp}\frac{vc_{\perp}}{c^2} & \overset{*}{W}\bar{Z} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overset{*}{W}\frac{\epsilon}{c^2} & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{c^2}{\overset{*}{c}{}_{\mathrm{L}}^{2}}\overset{*}{W}{}^2-1 & \beta'\overset{*}{W} & \overset{*}{W} & \overset{*}{W} & \overset{*}{W}{}^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta'\frac{v^2}{c_{\mathrm{L}}^2}\frac{c^2}{\epsilon} & \overset{*}{W} & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{2}{3}\bar{\Lambda}\frac{c^2}{\epsilon} & \cdot & \overset{*}{W}\bar{B} & \cdot & \frac{2}{3}\bar{\Lambda}\mathcal{M}\frac{c_{\mathrm{L}}}{v} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3}\Lambda^0\frac{c^2}{\epsilon} & \cdot & \cdot & \overset{*}{W}\bar{N} & \frac{1}{3}\Lambda^0\mathcal{N}\frac{c_{\mathrm{L}}}{v} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Lambda\frac{v^2}{\epsilon}\overset{*}{W} & \Lambda\frac{c_{\mathrm{L}}^2}{c^2} & \Lambda\mathcal{M}\frac{vc_{\mathrm{L}}}{c^2} & \Lambda\mathcal{N}\frac{vc_{\mathrm{L}}}{c^2} & \overset{*}{W}\bar{Z}
\end{array}\right)
\left(\begin{array}{c}
c\hat{u}^1 \\ \hat{\pi}^{12} \\ \hat{\pi}^{13} \\ \hat{q}^1 \\ \hline c\hat{u}^2 \\ \hat{\pi}^{22} \\ \hat{\pi}^{23} \\ \hat{q}^2 \\ \hline \hat{\rho} \\ c\hat{u}^3 \\ \hat{T} \\ \hat{\pi}^{33} \\ \hat{\pi} \\ \hat{q}^3
\end{array}\right) = 0 \qquad (4.13)
$$

The algebraic system of 14 homogeneous equations (4.13) has a non-trivial solution if and only if the appropriate determinant of this system vanishes, namely (because three equations are of second order in (4.13), our problem is really of order 17):

$$\Delta_{17}(W,\omega)=0, \qquad (W=\omega/\overset{\perp}{K}). \tag{4.14}$$

This is the characteristic equation, and its solutions (eigenvalues $W=W(\omega)$) define the dispersion dependence of the complete set of eigenmodes belonging to our system.

As is evident from (4.13) Δ_{17} is equal to the product of three lower order determinants $\Delta_{17}=\Delta_5\Delta_5\Delta_7$. Hence, instead of the dispersion equation (4.14), we need investigate only the two much simpler equations

$$\Delta_5=0 \qquad \text{and} \qquad \Delta_7=0, \tag{4.15}$$

the first corresponding to waves with transverse polarisation and the second to waves with longitudinal polarisation.

5. Transverse waves

5.1. General case (14- or 13-equation description)

The possible phase velocities $W=\omega/\overset{\perp}{K}$ with a transverse polarisation are given, using the first equation of (4.15) and (4.13) in dimensionless form, by the equation:

$$\Delta_5^{\mathrm{T}}\equiv \overset{*}{W}\bar{B}\begin{vmatrix} \dfrac{c^2}{\overset{*}{c}{}_\perp^2}\overset{*}{W}{}^2-1 & \overset{*}{W} & \overset{*}{W}{}^2 \\ \dfrac{1}{2}\bar{\Lambda}_\perp\dfrac{c^2}{\epsilon} & \overset{*}{W}\bar{B} & \dfrac{1}{2}\bar{\Lambda}_\perp\mathcal{M}_\perp\dfrac{c_\perp}{v} \\ \bar{\Lambda}_\perp\dfrac{v^2}{\epsilon}\overset{*}{W} & \bar{\Lambda}_\perp\mathcal{M}_\perp\dfrac{vc_\perp}{c^2} & \overset{*}{W}\bar{Z}\end{vmatrix}=0 \tag{5.1}$$

where

$$\frac{\overset{*}{c}{}_\perp^2}{c^2}=\frac{\mu}{\rho\epsilon+p}, \qquad \frac{c_\perp^2}{c^2}=\frac{\mu}{\rho\epsilon}, \qquad \xi\equiv\frac{c_\perp^2}{\overset{*}{c}{}_\perp^2}=1+\frac{p}{\rho\epsilon} \qquad \overset{*}{W}=\frac{W}{c},$$
$$\bar{\Lambda}_\perp=\frac{\bar{\lambda}}{\bar{\tau}}\frac{1}{\rho c_\perp^2}, \qquad \Lambda_\perp=\frac{\kappa}{\tau}\frac{1}{c_e\rho c_\perp^2}, \qquad \mathcal{M}_\perp=\rho Mc_\perp v, \qquad v^2=c_eT. \tag{5.2}$$

This equation can be reduced to the form

$$\Delta_5^{\mathrm{T}}\equiv \text{constant}\times\overset{*}{W}(\overset{*}{W}{}^2-\overset{*}{W}{}_{\mathrm{I}}^2)(\overset{*}{W}{}^2-\overset{*}{W}{}_{\mathrm{II}}^2)=0 \tag{5.3}$$

where

$$\overset{*}{W}{}_{\mathrm{I,II}}^2=\frac{1}{2\tilde{A}_\perp}[\tilde{B}_\perp\pm\sqrt{(\tilde{B}_\perp^2-4\tilde{A}_\perp\tilde{C}_\perp)}] \tag{5.4}$$

and†

$$\tilde{A}_\perp = \frac{c^2}{\overset{*}{c}{}_\perp^2}\left(\bar{B}\bar{Z} - \bar{B}\Lambda_\perp \frac{v^2}{\epsilon}\frac{\overset{*}{c}{}_\perp^2}{c^2}\right) \tag{5.5}$$

$$\tilde{B}_\perp = \bar{B}\bar{Z} + \frac{1}{2}\bar{\Lambda}_\perp \bar{Z}\frac{c^2}{\epsilon} + \frac{1}{2}\bar{\Lambda}_\perp \Lambda_\perp \left(\mathcal{M}^2 \xi - 2\mathcal{M}\frac{vc_\perp}{\epsilon}\right) \tag{5.6}$$

$$\tilde{C}_\perp = \frac{\overset{*}{c}{}_\perp^2}{c^2}\xi\left(\frac{1}{2}\bar{\Lambda}_\perp \Lambda_\perp \mathcal{M}_\perp^2\right). \tag{5.7}$$

This result is in agreement with the corresponding non-relativistic one because $\epsilon \to c^2$ and $\xi \to 1$ (see I, (4.5))‡. Only the last term on the right-hand side of equation (5.5) and (5.6) are new and purely relativistic and both disappear in the classical limit for which $\epsilon \to c^2$. The zero root of equation (5.3) has to be associated with the mass flow velocity, which was chosen in our case to be zero. Then we have two modes§ for transverse waves. Because $\overset{*}{W}{}_{\mathrm{I}}^2 \geq \overset{*}{W}{}_{\mathrm{II}}^2$, we will call the I-wave a fast transverse (or quasi-mechanical) wave and the II-wave a slow transverse (or quasi-thermal) wave, whose existence is mainly due to heat conduction, although we admit that such an intuitive distinction is not justifiable rigorously.

From the complex phase velocities $W_{\mathrm{I,II}}$, depending on the wave frequency through the expressions $\bar{B}(\omega)$ and $\bar{Z}(\omega)$ (see (4.12)), both the effective phase speed $W^+ = \omega/\mathrm{Re}\,\overset{\perp}{K}$ and the coefficient of absorption $\omega/\mathrm{Im}\,\overset{\perp}{K}$ can be deduced.

5.1.1. Limiting case when $\omega \to \infty$. The wavefront speed (signal speed) V for each wave mode can be found either from the complex W or from the real phase velocity W^+ as a limit:

$$V = \lim_{\omega\to\infty} W(\omega) = \lim_{\omega\to\infty} W[\bar{B}(\omega), \bar{Z}(\omega), \bar{N}(\omega)] \tag{5.8}$$

but $\bar{B}(\infty) = \bar{Z}(\infty) = \bar{N}(\infty) = 1$ by (4.12).

5.1.2. Limiting case when $\kappa, \bar{\lambda}, \lambda^0 \to 0$. If the thermal conductivity coefficient decreases to zero: $\kappa \to 0$ which also means $\tau \to 0$ by the third equation of (2.47), then because of equation (2.46) $q_\alpha \to 0$, which eliminates heat conduction (and also equation (2.46)) from the description. The same applies to $\bar{\lambda}$ and λ^0, so

$$\text{if}\begin{cases}\kappa \to 0\\ \bar{\lambda} \to 0\\ \lambda^0 \to 0\end{cases} \text{or equivalently}\begin{cases}|\bar{Z}| \to \infty\\ |\bar{B}| \to \infty\\ |\bar{N}| \to \infty\end{cases} \text{then}\begin{cases}q_\alpha \to 0\\ \pi_{\alpha\beta} \to 0\\ \pi \to 0.\end{cases} \tag{5.9}$$

† All the quantities occurring here, i.e. constants $\overset{*}{c}_{\mathrm{L}}, v, \epsilon, \Lambda_\perp, \ldots$ have to be evaluated in the initial unstrained state of the material: $T = T_0$ and $e_{\alpha\rho} = 0$ (see (3.1*a*)). Those coefficients (e.g. ϵ) depend however on the fixed reference temperature T_0 which may vary depending on the circumstances.

‡ In I $\overset{\perp}{W} \equiv W/c_\perp$, whereas here $\overset{*}{W} \equiv W/c = \overset{*}{c}_\perp \overset{\perp}{W}/c$, therefore $\tilde{A}_{(\mathrm{I})} = \overset{*}{c}{}_\perp^2 \tilde{A}_\perp/c^2, \tilde{B}_{(\mathrm{I})} = \tilde{B}_\perp$ and $\tilde{C}_{(\mathrm{I})} = c^2\tilde{C}_\perp/\overset{*}{c}{}_\perp^2$.

§ We call W^2 a mode, which means one wave propagating in the positive $(+W)$ and one in the negative sense $(-W)$, with the same speeds.

5.1.3. Limiting case when $\omega \to 0$. For this case

$$\lim_{\omega\to 0} \bar{Z} = \lim_{\omega\to 0}\left(1+\frac{\mathrm{i}}{\tau\omega}\right)\to 1+\mathrm{i}\infty, \qquad \text{or equivalently } |\bar{Z}|\to\infty,\ |\bar{B}|\to\infty,\ |\bar{N}|\to\infty \quad (5.10)$$

but these results are those for an adiabatic dissipation-free case, because the simultaneous transitions

$$\tau\to 0, \qquad \bar{\tau}\to 0, \qquad \tau^0\to 0,$$

also lead to the conditions (5.10), which, according to (5.9), lead to a cancellation of all dissipative fluxes and therefore to the adiabatic state.

5.2. *The adiabatic case* (*5-equation description*)

This case is included as a special, dissipation-free case of wave propagation when $\omega\to 0$ or by (5.10), when $|\bar{Z}|\to\infty$ and $|\bar{B}|\to\infty$ ($|\bar{N}|\to\infty$ need not be considered as transverse waves are independent of $\bar{N}$). Then from (5.5)–(5.7) and (5.4) it follows that

$$\overset{*}{W}{}^2_{\mathrm{I}} = \frac{\tilde{B}_\perp}{\overset{*}{A}_\perp} = \frac{c_\perp^2}{c^2} = \left(\frac{\mu}{\rho}\right)\frac{1}{\epsilon+(p/\rho)}, \qquad \overset{*}{W}{}^2_{\mathrm{II}} = 0, \qquad \left(\frac{\tilde{C}_\perp}{\overset{*}{A}_\perp} = 0\right). \quad (5.11)$$

5.3. *The case with shear viscosity only* (*10-equation description*)

This case with no heat conduction follows from (5.5)–(5.7) when we allow $|\bar{Z}|\to\infty$ (see (5.9)). Doing this, we obtain:

$$\overset{*}{W}{}^2_{\mathrm{I}} = \frac{\tilde{B}_\perp}{\overset{*}{A}_\perp} = \frac{\overset{*}{c}{}^2_\perp}{c^2}\left(1+\frac{1}{2}\bar{\Lambda}_\perp\frac{c^2}{\epsilon}\frac{1}{\bar{B}(\omega)}\right), \qquad \overset{*}{W}{}^2_{\mathrm{II}} = 0. \quad (5.12)$$

The wavefront speed (5.8) of the fast mode I (5.12) is

$$\overset{*}{V}{}^2_{\mathrm{I}} = |\overset{*}{W}_{\mathrm{I}}(\infty)|^2 = \frac{\mu}{\rho\epsilon+p}\left(1+\frac{1}{2}\bar{\Lambda}_\perp\frac{c^2}{\epsilon}\right) \quad \text{or} \quad \overset{*}{V}{}^2_{\mathrm{I}} = \frac{1}{\epsilon+(p/\rho)}\left(\frac{\mu}{\rho}+\frac{\mu'}{\rho\bar{\tau}}\right) \qquad (\bar{\lambda} = 2\mu'), \quad (5.13)$$

a value greater than that in the adiabatic case and this value is very sensitive to the shear viscosity relaxation time which, when set to zero (usually the case in conventional parabolic theories), results in infinite signal speed. It can be shown, and it is evident already from (5.13) that for any frequency $\omega > 0$, $W^+(\omega) \geq W_{\text{adiabatic}}$.

5.4. *The case without viscosity* (*8-equation description*)

This case with heat conduction is described if we allow $|\bar{B}|\to\infty$. On doing this, we obtain:

$$\overset{*}{W}{}^2_{\mathrm{I}} = \frac{\tilde{B}_\perp}{\overset{*}{A}_\perp} = \frac{\overset{*}{c}{}^2_\perp}{c^2}\bar{Z}\left(\bar{Z} - \Lambda_\perp\frac{v^2}{\epsilon}\frac{\overset{*}{c}{}^2_\perp}{c^2}\right)^{-1}; \qquad \overset{*}{W}{}^2_{\mathrm{II}} = 0,$$

$$\overset{*}{V}{}^2_{\mathrm{I}} = \mu\left((\rho\epsilon+p) - \frac{\kappa}{\tau}\frac{T}{c^2}\right)^{-1} > \frac{\mu}{\rho\epsilon+p}. \quad (5.14)$$

In the corresponding non-relativistic case (see I) the result is the same as in the adiabatic case, meaning that heat conduction does not produce dispersion in this approximation; however this is not the case for the relativistic description.

6. Longitudinal waves

6.1. General case (14-equation description)

The possible phase velocities $W = \omega/\overset{\perp}{K}$ with longitudinal polarisation are given by the following equation, using the second equation of (4.15) and equation (4.13) (in dimensionless form):

$$\Delta_7^{\mathrm{L}} \equiv \frac{\epsilon}{c^2}\overset{\mathrm{L}}{W}\frac{c^2}{\epsilon}\begin{vmatrix} \frac{\epsilon}{c^2}\left(\frac{c^2}{\overset{*}{c}{}_{\mathrm{L}}^2}\overset{\mathrm{L}}{W}{}^2-1\right) & \beta'\overset{\mathrm{L}}{W} & \overset{\mathrm{L}}{W} & \overset{\mathrm{L}}{W} & \overset{\mathrm{L}}{W}{}^2 \\ \beta'\frac{v^2}{c_{\mathrm{L}}^2} & \overset{\mathrm{L}}{W} & \cdot & \cdot & 1 \\ \frac{2}{3}\bar{\Lambda} & \cdot & \overset{\mathrm{L}}{W}\bar{B} & \cdot & \frac{2}{3}\bar{\Lambda}\mathcal{M}\frac{c_{\mathrm{L}}}{v} \\ \frac{1}{3}\Lambda^0 & \cdot & \cdot & \overset{\mathrm{L}}{W}\bar{N} & \frac{1}{3}\Lambda^0\mathcal{N}\frac{c_{\mathrm{L}}}{v} \\ \Lambda\frac{v^2}{c^2}\overset{\mathrm{L}}{W} & \Lambda\frac{c_{\mathrm{L}}^2}{c^2} & \Lambda\mathcal{M}\frac{vc_{\mathrm{L}}}{c^2} & \Lambda\mathcal{N}\frac{vc_{\mathrm{L}}}{c^2} & \overset{\mathrm{L}}{W}\bar{Z} \end{vmatrix} = 0 \tag{6.1}$$

where

$$\frac{\overset{*}{c}{}_{\mathrm{L}}^2}{c^2} = \frac{\lambda+2\mu}{\rho\epsilon+p}, \qquad \frac{c_{\mathrm{L}}^2}{c^2} = \frac{\lambda+2\mu}{\rho\epsilon}, \qquad \xi = \frac{c_{\mathrm{L}}^2}{\overset{*}{c}{}_{\mathrm{L}}^2} = 1+\frac{p}{\rho\epsilon}, \qquad \overset{\mathrm{L}}{W} = \frac{W}{c},$$

$$\delta = \beta'^2\frac{v^2}{c_{\mathrm{L}}^2}\frac{c^2}{\epsilon}, \qquad \beta' = \frac{\beta}{\rho c_e}, \qquad v^2 = c_e T, \qquad \mathcal{M} = \rho M c_{\mathrm{L}} v \tag{6.2}$$

$$\mathcal{N} = \rho N c_{\mathrm{L}} v, \qquad \bar{\Lambda} = \frac{\bar{\lambda}}{\bar{\tau}}\frac{1}{\rho c_{\mathrm{L}}^2}, \qquad \Lambda^0 = \frac{\lambda^0}{\tau^0}\frac{1}{\rho c_{\mathrm{L}}^2}, \qquad \Lambda = \frac{\kappa}{\tau}\frac{1}{\rho c_e c_{\mathrm{L}}^2}.$$

This equation can be reduced to the form

$$\Delta_7^{\mathrm{L}} \equiv \mathrm{constant} \times \overset{\mathrm{L}}{W}{}^3(\overset{\mathrm{L}}{W}{}^2 - \overset{\mathrm{L}}{W}{}_{\mathrm{I}}^2)(\overset{\mathrm{L}}{W}{}^2 - \overset{\mathrm{L}}{W}{}_{\mathrm{II}}^2) = 0 \tag{6.3}$$

where

$$\overset{\mathrm{L}}{W}{}_{\mathrm{I,II}}^2 = \frac{1}{2\tilde{A}}[\tilde{B} \pm \surd(\tilde{B}^2 - 4\tilde{A}\tilde{C})] \tag{6.4}$$

and (see the second footnote on p 1864)

$$\tilde{A} = \left(\frac{c^2}{\overset{*}{c}{}_{\mathrm{L}}^2}\right)\frac{\epsilon}{c^2}\bar{B}\bar{N}\left(\bar{Z} - \Lambda\frac{v^2}{\epsilon}\frac{\overset{*}{c}{}_{\mathrm{L}}^2}{c^2}\right), \tag{6.5}$$

$$\tilde{B} = \frac{\epsilon}{c^2}(1+\delta)\bar{B}\bar{N}\bar{Z} + \frac{2}{3}\left(\bar{\Lambda}\bar{N} + \frac{1}{2}\Lambda^0\bar{B}\right)\bar{Z} + \Lambda\frac{\epsilon}{c^2}\left[\xi - \frac{vc_{\mathrm{L}}}{c}\sqrt{\left(\frac{\delta}{\epsilon}\right)}\right]\bar{B}\bar{N}$$
$$+\frac{2}{3}\Lambda\bar{\Lambda}\frac{\epsilon}{c^2}\left(\mathcal{M}^2\xi - 2\mathcal{M}\frac{vc_{\mathrm{L}}}{\epsilon}\right)\bar{N} + \frac{1}{3}\Lambda^0\Lambda\frac{\epsilon}{c^2}\left(\mathcal{N}^2\xi - 2\mathcal{N}\frac{vc_{\mathrm{L}}}{\epsilon}\right)\bar{B}, \tag{6.6}$$

$$\tilde{C}=\left(\frac{\overset{*}{c}{}_{L}^{2}}{c^{2}}\right)\xi\frac{\epsilon}{c^{2}}\left\{\Lambda\bar{B}\bar{N}+\frac{2}{3}\Lambda\bar{\Lambda}\left[\mathcal{M}^{2}+\left(\frac{c}{\sqrt{\epsilon}}-\mathcal{M}\sqrt{\delta}\right)^{2}\right]\bar{N}\right.$$
$$\left.+\frac{1}{3}\Lambda\Lambda^{0}\left[\mathcal{N}^{2}+\left(\frac{c}{\sqrt{\epsilon}}-\mathcal{N}\sqrt{\delta}\right)^{2}\right]\bar{B}+\frac{2}{9}\Lambda\Lambda^{0}\bar{\Lambda}\frac{c^{2}}{\epsilon}(\mathcal{M}-\mathcal{N})^{2}\right\}. \tag{6.7}$$

This result is also in agreement with the corresponding non-relativistic one (see I, equations (5.5)–(5.7)) except for several new, purely relativistic terms, which all disappear in the classical limit in which $\epsilon \to c^2$ (see (A.7)).

From (6.3) we see that there are once again two longitudinal modes, and because of $\overset{L}{W}{}_{\rm I}^2 \geq \overset{L}{W}{}_{\rm II}^2$ (from (6.4)), we call the I-wave a fast longitudinal wave or 'quasi-mechanical wave' (or sound wave) and the II-wave a slow wave or 'quasi-thermal wave'. Let us turn to some special cases.

6.2. *The adiabatic case (5-equation description)*

This case can be obtained according to (5.10) as a limiting case for $|\bar{B}|\to\infty$, $|\bar{N}|\to\infty$ and $|\bar{Z}|\to\infty$. Taking those limits, using (6.4)–(6.7), we obtain

$$\overset{L}{W}{}_{\rm I}^{2}=\frac{\overset{*}{c}{}_{L}^{2}}{c^{2}}(1+\delta)=\left(\frac{\lambda+2\mu}{\rho}\right)\frac{1}{\epsilon+(p/\rho)}+(\beta'^{2}c_{e}T)\frac{1}{\epsilon+(p/\rho)};$$
$$\overset{L}{W}{}_{\rm II}^{2}=0 \qquad \left(\text{as}\ \frac{\tilde{C}}{\overset{*}{A}}=0\right) \tag{6.8}$$

where δ is the so called thermoelastic (dimensionless) coupling coefficient.

In the formula (6.8), valid for thermoelastic continua (see equation (2.53)) there is a factor $[\epsilon+(p/\rho)]^{-1}$ (where $p=A$ is an initial pressure) which is well known from the relativistic description of a fluid, namely of the ideal monatomic gas (where $p=\rho RT$) for which the phase speed is that given by Synge's (1957) formula (316) (cf Kranyš 1972, equation (2.32))†

$$W_{\rm I}^{2}=\frac{c_p}{c_v}RT\frac{1}{\tilde{\epsilon}+(p/\rho)};\qquad \left(c_p=-R\gamma^{2}\frac{{\rm d}G}{{\rm d}\gamma}=c_v+R,\quad \tilde{\epsilon}=c^{2}\left(G-\frac{1}{\gamma}\right),\quad p=c^{2}\frac{\rho}{\gamma}\right). \tag{6.8a}$$

The simpler result $\overset{L}{W}{}_{\rm L}^{2}=\overset{*}{c}{}_{L}^{2}/c^{2}$ follows from (6.8) for the 'uncoupled' case when the stress is not directly influenced by heating ($\beta'=0$ and $\delta=0$). This simpler result is identical with that of Carter (1973) for the perfect solid under high pressure based on sophisticated considerations. The result (6.8) with $\beta'=0$ and $p=0$ was given by Synge (1959).

In the non-relativistic limit ($kT_0 \ll mc^2$ or $\gamma\to\infty$; see (A.7)) the factor

$$\epsilon+\frac{p}{\rho}=c^{2}\left(1+\frac{\Delta\epsilon}{c^{2}}+\frac{p}{\rho c^{2}}\right)$$

(writing $\epsilon=c^2+\Delta\epsilon$) reduces to c^2 and the formula (6.8) *formally* coincides with the one well known from classical physics, where the residual pressure p does not appear.

† For a monatomic relativistic gas (with three degrees of freedom) we have, for example $\tilde{\epsilon}=c^2[G(\gamma)-(1/\gamma)]$, where $G=K_3/K_2$ and $K_n(\gamma)$ are the Kelvin-Bessel functions; so that $\lim_{\gamma\to\infty}\tilde{\epsilon}\approx c^2[1+\frac{3}{2}(1/\gamma)]$ and $\lim_{\gamma\to 0}\tilde{\epsilon}\approx c^2(3/\gamma)=3(kT/m)=3RT$. ($\gamma$ is defined by (A.6).)

6.3. The case when dissipation is due only to viscosity (11-equation description)

This case follows from the general formulae (6.4)–(6.7) with $|\bar{Z}| \to \infty$ (i.e. elimination of heat flux; see (5.9)) which leads to the expressions

$$\overset{L}{W}{}^2_{\mathrm{I}} = \frac{\lambda + 2\mu}{\rho\epsilon + p}\left((1+\delta) + \frac{2}{3}\bar{\Lambda}\frac{c^2}{\epsilon}\frac{1}{\bar{B}(\omega)} + \frac{1}{3}\Lambda^0\frac{c^2}{\epsilon}\frac{1}{\bar{N}(\omega)}\right), \qquad \overset{L}{W}{}^2_{\mathrm{II}} = 0 \tag{6.9}$$

telling us that the slow, quasi-thermal mode disappears and the main acoustical mode survives giving, as a result of bulk and shear viscosity, a higher phase velocity than the adiabatic sound speed.

If we eliminate either bulk viscosity ($|\bar{N}| \to \infty$) (resulting in a 13-equation description) or shear viscosity ($|\bar{B}| \to \infty$) (leading to a 9-equation description) both propagation modes survive, and only the dispersion curves are modified accordingly.

6.4. The case when dissipation is due to heat conduction only (8-equation description)

This case follows from (6.4)–(6.7) with $|\bar{B}| \to \infty$ and $|\bar{N}| \to \infty$ leading to the two non-trivial modes

$$\overset{L}{W}{}^2_{\mathrm{I,II}} = \frac{\overset{*}{c}{}^2_{\mathrm{L}}}{c^2}\frac{1}{2\phi}\left\{(1+\delta) + \frac{\Lambda}{\bar{Z}} \pm \sqrt{\left[\left((1+\delta) + \frac{\Lambda}{\bar{Z}}\psi\right)^2 - 4\frac{\Lambda}{\bar{Z}}\xi\phi\right]}\right\} \tag{6.10}$$

where

$$\phi \equiv 1 - \frac{\Lambda}{\bar{Z}}\frac{\overset{*}{c}{}^2}{c^2}\frac{v^2}{\epsilon}, \qquad \psi \equiv \xi - \frac{v c_{\mathrm{L}}}{c}\sqrt{\frac{\delta}{\epsilon}} \tag{6.11}$$

which, in the non-relativistic case (i.e. $\epsilon \to c^2$, (A.7) and $v^2 \ll c^2$, $c_{\mathrm{L}}^2 \ll c^2$), change (due to $\psi \to 1$, $\phi \to 1$ and $\xi \to 1$) to equation (5.11) in I.

7. The hyperbolicity of the theory

The requirement that our system of 14 partial differential equations (3.8)–(3.13) be hyperbolic can be formulated (Courant and Hilbert 1966, §§ 3.3, 3.6) in the following way. If the characteristic equation of the system under consideration, which in our case is the characteristic polynomial $\Delta_{17} = (\Delta_5^{\mathrm{T}})^2\Delta_7^{\mathrm{L}}$ (Δ_5^{T} and Δ_7^{L} being given by (5.3) and (6.3) respectively) in the limit $\omega \to \infty$:

$$\lim_{\omega\to\infty}\Delta_{17} \equiv \text{constant} \times \lim_{\omega\to\infty}(\overset{*}{W}{}^2 - \overset{*}{W}{}^2_{\mathrm{I}})^2(\overset{*}{W}{}^2 - \overset{*}{W}{}^2_{\mathrm{II}})^2\overset{*}{W}{}^2(\overset{L}{W}{}^2 - \overset{L}{W}{}^2_{\mathrm{I}})(\overset{L}{W}{}^2 - \overset{L}{W}{}^2_{\mathrm{II}})\overset{L}{W}{}^3 = 0 \tag{7.1}$$

possesses *only real and finite solutions for all roots*, then the system is hyperbolic. First, five zero roots $W^5 = 0$, as well as $\overset{*}{W}{}^2_{\mathrm{I,II}}$ (given by (5.4) and $\overset{L}{W}{}^2_{\mathrm{I,II}}$ (given by (6.4)) fulfill this condition of reality and finiteness as long as $\bar{\Lambda} < \infty$, $\Lambda^0 < \infty$ and $\Lambda < \infty$. This is fulfilled if and only if we have simultaneously (cf (5.2) and (6.2)):

$$\bar{\tau} > 0, \qquad \tau^0 > 0, \qquad \tau > 0, \tag{7.2}$$

because all the coefficients in (5.4) and (6.4) are real.

If only one of the relaxation constants is equal to zero then all the propagation wave modes have infinite wavefront speeds. We notice that the relaxation constants (7.2) are *definitely responsible* for the finiteness of wavefront speeds and therefore for guaranteeing the *causality principle* and *can therefore not be neglected.*

8. Comparison with other theories and results

This will help us to determine the differences and weaknesses of those particular theories. As an example, let us compare the parabolic equation for longitudinal strain waves based on the relativistic Kelvin–Voigt stress–strain relation proposed by Maugin (1974a, equation (4.3) in which we disregarded driving gravitational field) with our corresponding results in § 6.3. The dispersion law for propagation modes in Maugin's case ($q^\alpha = 0$, $\delta = 0$, and $p = 0$) can be deduced from (6.9) putting $\tau^0 = \bar{\tau} = N = M = 0$. Keeping in mind that (by virtue of (4.12) and (6.2))

$$\lim_{\bar{\tau}\to 0} \frac{\bar{\Lambda}}{\bar{B}} = -\mathrm{i}\omega(\bar{\Lambda}\bar{\tau}), \qquad \lim_{\tau^0\to 0} \frac{\Lambda^0}{\bar{N}} = -\mathrm{i}\omega(\Lambda^0\tau^0), \tag{8.1}$$

we easily find that:

$$\overset{\mathrm{L}}{W}{}^2_{\mathrm{I}} = \frac{1}{\epsilon}\left[\left(\frac{\lambda + 2\mu}{\rho}\right) - \frac{2}{3}\mathrm{i}\omega\frac{1}{\rho}\left(\bar{\lambda} + \frac{1}{2}\lambda^0\right)\right], \qquad \overset{\mathrm{L}}{W}{}^2_{\mathrm{II}} = 0. \tag{8.2}$$

This formula is valid if heat conduction can be neglected and if one is interested only in the dispersion dependence for not too high a frequency range, i.e. $\bar{\tau}\omega \ll 1$, $\tau^0\omega \ll 1$. Of course the limit (5.8), i.e. $\omega \to \infty$ applied to the expression (8.2) shows that the speed of the wavefront is infinite, proving the non-hyperbolicity of utilised theory. In the classical limit $\epsilon \to c^2$ we obtain the well known formulae (in Maugin's formula (4.3) the factor $1/\epsilon$ is missing). Yet one has to realise that besides the fast, longitudinally polarised wave, there is also a fast, transverse wave (5.12) and that, due to our neglect of heat conduction, the slow II-modes did not survive in this particular case.

Therefore, in spite of some weaknesses in Maugin's results, we must confirm his criticism of Weber's (1961) equation (8.34) which is hyperbolic but which leads to an incorrect dispersion dependence (see I, § 7.1).

Let us consider again Maugin's case, but with $\pi^{\alpha\beta} = 0(\bar{\Lambda} = 0)$ to show how the terms responsible for dissipation give rise to a phase and signal speed which is higher than is the adiabatic speed. The equations (3.9)–(3.11) governing this case will contain only $\overset{\perp}{\partial}_3$ (of all $\overset{\perp}{\partial}_\alpha$) and with $\overset{\perp}{\partial}_3\overset{\perp}{\partial}{}^3 = \Delta$, $cu_3 = u$, and $c\mathscr{D}u = \dot{u}$ they will read

$$\frac{\rho\epsilon}{c^2}\ddot{u} - (2\mu + \lambda)\Delta u + \overset{\perp}{\partial}_3\dot{\pi} = 0, \qquad T = \text{constant} \tag{8.3}$$

$$\pi + \tau^0\dot{\pi} + \tfrac{1}{3}\lambda^0\overset{\perp}{\partial}{}^3 u = 0. \tag{8.4}$$

Applying $c\mathscr{D}\overset{\perp}{\partial}_3$ derivative on equation (8.4) and then expressing $\overset{\perp}{\partial}_3\dot{\pi}$ by the use of (8.3) one obtains

$$\left((2\mu + \lambda)\Delta u - \frac{\rho\epsilon}{c^2}\ddot{u}\right) + \tau^0 c\mathscr{D}\left[\left(2\mu + \lambda + \frac{1}{3}\frac{\lambda^0}{\tau^0}\right)\Delta u - \frac{\rho\epsilon}{c^2}\ddot{u}\right] = 0 \tag{8.5}$$

(and a similar equation for π, if desired). As in this equation only one unknown appears (i.e. u) it is completely independent of the remaining unknowns (π, T, ρ) and one may conclude that

$$\frac{W_{eq}^2}{c^2} = \frac{2\mu + \lambda}{\rho\epsilon} \tag{8.6}$$

in the adiabatic case, i.e. if $\lambda^0 = 0$, $\tau^0 = 0$ (by the first equation of (2.47))

$$\frac{V^2}{c^2} = \frac{1}{\epsilon}\left[\left(\frac{2\mu + \lambda}{\rho}\right) + \frac{1}{3}\frac{\lambda^0}{\tau^0}\frac{1}{\rho}\right] \geq \frac{W_{eq}^2}{c^2} \tag{8.7}$$

(by (2.39) and (2.47)) if $\lambda^0 > 0$. The expression (8.7) is obtained by comparing the principal part (i.e. the part containing the highest order derivatives) with the wave equation, being in conformity with (6.9), and representing the square of the signal speed which is also the speed of propagation of the characteristic surfaces associated with equation (8.5). The dispersion dependence $W^+(\omega)$ then satisfies the inequality:

$$V \geq W^+(\omega) \geq W_{eq} \qquad (\infty \geq \omega \geq 0) \tag{8.8}$$

but in the case of Weber's equation we have $V = W_{eq} \geq W^+(\omega)$.

If our theory of continua is specialised to a gas it is not identical with the linearised form of Marle's (1969) theory, as it is evident on comparison of the energy–momentum tensors for both theories. We utilised Eckart's tensor in conformity with Israel's (1976) proposition in order to obtain the governing equation in the same form as Müller (1966) and Israel did, i.e. in a form which is slightly more symmetrical than in Marle's formalism. However, at present we are not able to say which choice is more realistic.

As a peripheral result of our study of modes in dissipative elastic continua, we have obtained a more general expression for the adiabtic mode than that contained in Carter's (1973) work.

9. Correspondence between the coefficients for different phase states

We can hardly expect that a conventional material can exist in the solid state for relativistic temperatures ($kT_0 \gg mc^2$ or even $kT_0 \approx mc^2$) but rather it must exist in the gaseous state (disregarding dissociation of molecules) which has no rigidity ($\bar{\theta}^{\alpha\beta} = 0$) and therefore $\mu = 0$. So one can expect, purely formally, that solid matter, upon increase in temperautre of the reference state T_0, will have a decreasing rigidity modulus μ which finally becomes zero at temperatures $T_0 \geq \overset{*}{T}_0$ where the matter will exist only in the fluid state or perhaps even in the gaseous state. We will not enter into the physics of phase transitions but we will show that the form (structure) of the phase velocity or pressure in terms of coefficients like $\lambda, \beta, \ldots$, i.e. $W(\lambda, \beta, \ldots)$ or $W(\chi[\lambda, \beta, \ldots], \nu[\lambda, \beta, \ldots])$ *persists* for the different phase states of the matter in question: and hence we can follow each coefficient and its sometimes multiple manifestations for the solid, liquid or gaseous state.

The coefficients of isothermal compression χ and isometric thermal tension ν may be defined as:

$$\chi \equiv -\left(\frac{\partial p}{\partial e^{\gamma}_{\gamma}}\right)_{T_0} = -\left(\frac{\partial p}{\partial \rho}\frac{\mathscr{D}\rho}{\mathscr{D}e^{\gamma}_{\gamma}}\right)_{T_0} = \rho_0\left(\frac{\partial p}{\partial \rho}\right)_{T_0} \qquad \text{by (2.6) and (2.11);}$$
$$\nu \equiv \left(\frac{\partial p}{\partial T}\right)_{e_0} = \left(\frac{\partial p}{\partial T}\right)_{\rho_0}. \tag{9.1}$$

Let us evaluate these terms for a Hookean thermoelastic solid (2.53), for a Van der Waals fluid, i.e. when $p = [\rho RT/(1-b\rho)] - a\rho^2$, and for an ideal gas, i.e. when $p = \rho RT$. We obtain:

$$\chi_{\text{solid}} = \left(\lambda + \frac{2}{3}\mu\right), \qquad \nu_{\text{solid}} = \beta,$$
$$\chi_{\text{fluid}} = \rho_0\left(\frac{RT_0}{(1-b\rho)^2} - 2\rho_0 a\right) = p_0 - a\rho_0^2, \qquad \nu_{\text{fluid}} = \frac{\rho_0 R}{1-b\rho}, \tag{9.2}$$
$$\chi_{\text{gas}} = \rho_0 RT_0 = p_0, \qquad \nu_{\text{gas}} = \rho_0 R,$$
$$(c_p - c_v)_{\text{fluid}} = -T\frac{(\partial p/\partial T)_v^2}{(\partial p/\partial v)_T} = \frac{T\nu^2}{\rho\chi} = \delta c_v, \qquad (c_p - c_v)_{\text{gas}} = \frac{T\nu^2}{\rho\chi} = R = \delta c_v.$$

Expanding $p = p(\rho, T)$ about the reference point (ρ_0, T_0) in order to find the linearised form of p consonant to that of (2.18a)

$$p(T_0 + \Delta T, \rho_0 + \Delta\rho) = p_0 + \nu\Delta T + \chi\frac{1}{\rho_0}\Delta\rho, \qquad p_0 = p(\rho_0, T_0) \tag{9.3}$$

and passing from the variable ρ to e^{γ}_{γ}, by virtue of the second equation of (2.6): $-\Delta\rho = \rho\Delta e^{\gamma}_{\gamma} = \rho(e^{\gamma}_{\gamma} - 0)$, equation (9.3) reads

$$p(T, e^{\gamma}_{\gamma}) = p_0 + \nu(T - T_0) - \chi e^{\gamma}_{\gamma}, \tag{9.4}$$

which is formally identical with the first equation of (2.53). By means of (9.4) we obtain the general expression for the fluid part of the pressure tensor $p\overset{\perp}{g}{}^{\alpha\beta} = \theta^{\alpha\beta} - \bar{\theta}^{\alpha\beta}$ for solid, liquid or gaseous state of matter depending only on which values of the coëfficients χ and ν obtained from (9.2) are used.

Now by virtue of the sixth equation of (6.2), equation (6.8), with $\mu \to 0$ (i.e. also $c_e \to c_v$), takes the form (we have dropped subscript 0):

$$\left(\epsilon + \frac{p}{\rho}\right)\overset{\perp}{W}{}^2_{\text{I}} = \frac{1}{\rho}\left(\lambda + \frac{\beta^2 T}{\beta c_e}\right) = \frac{\chi}{\rho}\left(1 + \frac{\nu^2 T}{\rho c_v \chi}\right) = \frac{\chi}{\rho}(1+\delta). \tag{9.5}$$

On the other hand Synge's formula for the ideal gas (6.8a) taking into account equation (9.2) takes the form

$$\left(\tilde{\epsilon} + \frac{p}{\rho}\right)W^2_{\text{I}} = \frac{c_p}{c_v}RT = RT\left(1 + \frac{R}{c_v}\right) = \frac{\chi}{\rho}(1+\delta), \tag{9.6}$$

which coincides with that of (9.5). This natural correspondence of form (CF) of expression (9.5) with (9.6) or (9.4) with the first equation of (2.53) for different phase states would not be possible, for example, if the thermoelastic effect were disregarded, as in Carter (1973).

It is also easy to follow CF in the case of other coefficients characterising dissipation phenomena. We see, e.g., that $\bar{\lambda} = 2\mu'$ and $\frac{1}{3}\lambda^0 = \lambda' + \frac{2}{3}\mu'$ also have the meaning of shear and bulk viscosity coefficients respectively in the case of a fluid, in which case the explicit form (derived from kinetic theory) can be found in the literature.

The demonstrated existence of CF together with (9.2) can be used for a rough estimate of values as $\chi_{\text{solid}}, \nu_{\text{solid}}, \ldots$ in terms of $\chi_{\text{fluid}}, \nu_{\text{fluid}}, \ldots$ or even $\chi_{\text{gas}}, \nu_{\text{gas}}, \ldots$ as these latter are better known in the high temperature region.

9.1. Extraction of fluid-continua normal modes

If one makes, in our non-fluid continua theory the substitutions (cf equation (9.2))

$$\mu \to 0 \quad (\text{i.e. } \bar{\theta}^{\alpha\beta} = 0), \qquad \lambda \to \chi_{\text{fluid}}, \qquad \beta \to \nu_{\text{fluid}}, \qquad \left(\text{and } \mathscr{D}e^{\gamma}_{\gamma} = -\frac{\mathscr{D}\rho}{\rho}\right), \tag{9.7}$$

then this theory reduces itself to Müller's or Israel's original fluid theory with the equation of state $p = p(T, \rho)$ in the linearised form

$$p = p_0 + \nu_{\text{fluid}}(T - T_0) + \chi_{\text{fluid}} \frac{1}{\rho_0}(\rho - \rho_0), \tag{9.8}$$

and the results of §§ 5 and 6 valid for linearised elasticity theory reduce to the ones for linearised fluid. In that case the adaptation of results of § 6 is trivial; but for transverse polarised waves (§ 6) where $\mu \to 0$ requires that $c_\perp^2 = 0$ we must introduce the (primed) quantities which are independent of μ, in order to visualise the dependence of our expressions for μ:

$$\begin{aligned} &c_\perp'^2 = \frac{1}{\mu} c_\perp^2, \qquad \overset{*}{c}{}_\perp'^2 = \frac{1}{\mu} \overset{*}{c}{}^2, \qquad \bar{\Lambda}_\perp' = \mu \bar{\Lambda}_\perp, \\ &\Lambda_\perp' = \mu \Lambda_\perp, \qquad \mathscr{M}_\perp'^2 = \frac{1}{\mu} \mathscr{M}_\perp^2. \end{aligned} \tag{9.9}$$

Upon inserting these expressions into equations (5.4)–(5.7) and allowing $\mu \to 0$ we easily obtain

$$\overset{*}{W}{}_{\mathrm{I}}^2 = \frac{\tilde{B}}{\tilde{A}} = \frac{\overset{*}{c}{}'^2}{c^2} \frac{\bar{\Lambda}_\perp'}{2\bar{B}} \left[\bar{Z} \frac{c^2}{\epsilon} + \Lambda_\perp' \left(\xi \mathscr{M}'^2 - 2\mathscr{M}' \frac{c_\perp' v}{\epsilon} \right) \right] \left(\bar{Z} - \Lambda_\perp' \frac{\overset{*}{c}{}'^2}{c^2} \frac{v^2}{\epsilon} \right)^{-1};$$

and

$$\frac{\tilde{C}}{\tilde{A}} = 0, \quad (\text{or } \overset{*}{W}{}_{\mathrm{II}}^2 = 0). \tag{9.10}$$

We see that one of the transverse modes has disappeared as is expected for fluid continua. The result (9.10) is formally very similar to that in Kranyš (1976, equation (2.3)) but not identical. Allowing $|\bar{Z}| \to \infty$ in the expression (9.10) we obtain

$$\overset{*}{W}{}_{\mathrm{I}}^2 = \frac{\overset{*}{c}{}'^2 \bar{\Lambda}_\perp'}{c^2} \frac{c^2}{\epsilon} \frac{1}{2\bar{B}} = \frac{\overset{*}{c}{}'^2 \bar{\Lambda}_\perp'}{\epsilon} \frac{1}{2\bar{B}}, \tag{9.11}$$

which conforms with equation (2.3) mentioned above and which shows that the difference we referred to must have its origin in the heat conduction treatment.

10. Conclusions

A phenomenological general relativistic theory for a dissipative elastic solid whose equations form a hyperbolic system is proposed. The non-stationary transport equations for dissipative fluxes containing new cross-effect terms, as required for compatibility with the entropy principle expressed by the balance equation with all second-order terms, have been adopted in order to guarantee physical causality and the possibility of describing fast, transient processes. In the adaptation of Muller's or Israel's theory, which is a fluid theory, to elasticity, the principal step was the inclusion of only the purely irreversible (or dissipative) part of the stress tensor in the transport equations, which is consistent with non-equilibrium irreversible thermodynamics. There has never been a theory of elasticity, before this one, containing a stress–strain relation with a stress relaxation compatible with thermodynamics. The theory formed from the system of 14 partial differential equations (in the case of special relativity), of total order 17, is hyperbolic. Five new transport coefficients appear in the transport equations, in contrast to conventional parabolic theories; however, three of them (relaxation times) have been investigated previously, in connection with some simpler constitutive equations.

The complete system of special relativistic propagation modes has been determined from the 14 linearised equations. There are four mutually distinct non-trivial propagation modes, two for longitudinal waves, and two for transverse waves. Those modes for relativistic dissipative solids are predicted here for the first time, and their non-relativistic limits agree with those deduced by Kranyš (1977). The wavefront of each mode propagates with a finite velocity, affording direct proof of the hyperbolicity of the theory. The wavefront speeds of the modes (being always higher than those for dissipation-free propagation), represent the speed of propagation of the characteristic surfaces on which a discontinuity of some quantities can occur, and therefore represent weak shock wave speeds, or, more exactly, shock precursor speeds. However the values of wavefront speeds cannot be evaluated numerically if we base ourselves exclusively on our phenomenological theory. So we gave at least some suggestions on how to obtain rough asymptotic expressions for the phenomenological coefficients.

As the relativistic effects are small in elastic bodies in the conditions presently accessible, there is no literature about the experimental investigation of such effects to date, so that direct comparison of the present theory with experiments is impossible. However in the case of classical theory there exists a great deal of literature on this subject and the theory proposed in I, which appears as a non-relativistic limit of that proposed here, is at least in qualitative agreement with experiments (see I). It would be desirable to have experimental values of the five newly introduced constants for a quantitative comparison between theory and experiment, for various solids.

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Appendix 1. Some auxiliary formulae and definitions

In deriving and handling the constitutive equations the Lie derivative $\mathscr{L}$ with respect of u^α has been used (see, e.g. Schouten 1954). It is also possible to work with even more suitable convected derivatives (see Carter and Quintana 1972†). For example

$$\mathscr{L}T_{\alpha\beta} \equiv \mathscr{D}T_{\alpha\beta} + T_{\alpha\sigma}\nabla_\beta u^\sigma + T_{\sigma\beta}\nabla_\alpha u^\sigma; \qquad \mathscr{L}s \equiv \mathscr{D}s; \tag{A.1}$$

$$\mathscr{L}w^\kappa \equiv \mathscr{D}w^\kappa - w^\mu\nabla_\mu u^\kappa; \qquad \mathscr{L}w_\lambda \equiv \mathscr{D}w_\lambda + w_\mu\nabla_\lambda u^\mu \tag{A.2}$$

By (A.2):

$$\mathscr{L}u^\alpha \equiv \mathscr{D}u^\alpha - u^\mu\nabla_\mu u^\alpha = 0; \qquad \mathscr{L}u_\alpha \equiv \mathscr{D}u_\alpha + u_\mu\nabla_\alpha u^\mu = \mathscr{D}u_\alpha \tag{A.2a}$$

and then by the first equation of (A.1):

$$\mathscr{L}g_{\alpha\beta} \equiv \mathscr{D}g_{\alpha\beta} + g_{\alpha\sigma}\nabla_\beta u^\sigma + g_{\alpha\beta}\nabla_\alpha u^\sigma = 0 + \nabla_\beta u_\alpha + \nabla_\alpha u_\beta = 2\nabla_{(\alpha}u_{\beta)} \tag{A.3}$$

and therefore

$$\begin{aligned}\overset{\perp}{\mathscr{L}}g_{\alpha\beta} &\equiv \mathscr{L}(g_{\alpha\beta} + u_\alpha u_\beta)\\ &= \mathscr{L}g_{\alpha\beta} + u_\beta\mathscr{L}u_\alpha + u_\alpha\mathscr{L}u_\beta = 2\nabla l_{(\alpha}u_{\beta)} + 2u_{(\alpha}\mathscr{D}u_{\beta)}\\ &= 2\overset{\perp}{\nabla}_{(\alpha}u_{\beta)} = 2[\nabla_{(\alpha}u_{\beta)}]_\perp \end{aligned} \tag{A.4}$$

by the third equation of (2.5) and the second and third equations of (2.3). From (A.4) by definition (2.3), it follows that

$$(\tfrac{1}{2}\overset{\perp}{\mathscr{L}}g_{\alpha\beta})u^\alpha = [\nabla_{(\alpha}u_{\beta)}]_\perp u^\alpha = 0. \tag{A.5}$$

Appendix 2. Remark about the classical and ultra-relativistic limit

To find the classical (i.e. non-relativistic) or ultra-relativistic limit for the phenomenological coefficient is not possible unless those coefficients are sufficiently specified or if an estimate of their size can be made. For example, for 'not too low' and 'not too high' temperature, according to the equipartition theorem, the internal energy of a solid can be estimated as $E \simeq nmc^2 + n3kT$ so that the specific internal energy is $\epsilon = E/\rho \simeq c^2[1 + 3(kT/mc^2)]$ (see footnote to § 6.2). By the classical (or low temperature) limit we understand the case when the effective dimensionless temperature γ^{-1} is low, i.e.

$$\frac{1}{\gamma} \equiv \frac{kT}{mc^2} \ll 1; \tag{A.6}$$

in that case ϵ becomes the relativistic rest energy and we always obtain:

$$\lim_{\gamma\to\infty} \epsilon = c^2. \tag{A.7}$$

† In the part of the text dealing with the constitutive equations we operate with $\mathscr{L}$ derivatives which, when applied on tensors of O(1) like $e_{\alpha\beta}$ lead to contributions of O(2) which terms can often be neglected in comparison with other lower order terms in the governing equations.

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